Power Round: Gambling with Martingales

PVMT 2026

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§1 Rules

Welcome to the Power Round! This is a **proof-based** component of PVMT, and as such, all solutions in your submission must demonstrate conceptual understanding and mathematical reasoning. Claims with no attempt at a proof will not be given credit. Other important information:

- You are **not** allowed to consult team-external people, AoPS forums, any form of AI, search up the problem statements, write computer code, or access any other outside interactive informational resources. The only exceptions to this are calculators and standard mathematical reference sites for definitions (i.e. Wikipedia, PlanetMath).

  Any team found to violate this will have their score invalidated; please don't cheat on a low-stakes part of low-stakes contest.
- Major theorems or facts (e.g. the Fundamental Theorem of Arithmetic) can be stated without justification if properly named/cited in your solution.
- For a certain problem, you are allowed to cite all information previous to it in the power round packet in your solution, though **you are not allowed to cite any later results**.
- Most of your time is recommended to be spent fully understanding the material, it is strongly not recommend to directly rush into the problems.
- There will be **generous** partial credit, please do not be afraid to submit any work you have! Conjectures, simplified cases, etc. are all available for credit.
- Don't make stuff up in your solutions, you will receive more partial credit for having correct vague intuition, than having something that *looks* rigorous but is false.
- **A solution without using section-specific theory will receive partial but not full credit.**
- For the later problems in Section 6, there may be some *intermediate* results that just require clever combinatorics, although the full solution will require martingales. Any work for these intermediate results is available for credit, even without a full solution.

There are **10 problems** and **80 points** available in total. Problems are located at the end of Section 3, Section 4.1, Section 4.2, and Section 6. The problems have different weights; greater weights mean greater opportunity for partial credit!

Submissions will be submitted through email or the website, and can either be computer-formatted (ie. g-docs, LaTeX) or scanned handwritten pages (though please try to keep it legible!) We encourage you to email us at falconsdomath@gmail.com for clarifications.

§2 Background and definitions

Mathematics is full of sequences that "drift": a falling stone speeds up, a population grows, a stock's dividends hopefully rises on average. Martingale theory is the study of sequences that have *no drift at all* — processes that are perpetually fair.

Martingales have become one of the most powerful tools in all of probability, with applications ranging from card games and random walks to the Black-Scholes option pricing formula and the proof of the Riemann hypothesis (in function field settings).

Definition 2.1. We work on a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$:

- Ω is the **sample space**: the set of all possible outcomes of our random experiment.
- $\mathcal{F}$ is a **σ-algebra** (or σ-field): a collection of subsets of Ω called *events*.
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a **probability measure** satisfying $\mathbb{P}(\Omega) = 1$ and countable additivity. (For this power round, you don't have to worry about checking countable additivity.)

Definition 2.2. We say an event holds **almost surely** (abbreviated **a.s.**) if it occurs with probability 1.

Definition 2.3. A σ-algebra $\mathcal{F}$ is a collection of subsets of Ω satisfying:

(i) $\Omega \in \mathcal{F}$,
(ii) if $A \in \mathcal{F}$ then the complement $A^c \in \mathcal{F}$,
(iii) if $A_1, A_2, \ldots \in \mathcal{F}$ then $A_1 \cup A_2 \cup \cdots \in \mathcal{F}$.

A σ-algebra is precisely the collection of questions you are allowed to ask about the "world" of the experiment. Each set $A \in \mathcal{F}$ corresponds to the yes/no question "did outcome A occur?". The closure properties just say that if you can ask a question, you can also ask its negation and the union of any list of such questions – a very natural requirement.

Example 2.4

Let's look at the world of an experiment where we flip a fair coin twice. The sample space will have four elementary outcomes:

$$\Omega = \{HH, HT, TH, TT\}.$$

Now let's look at our σ-algebra ($\mathcal{F}$). If we want to be able to ask any possible question about these two flips, our σ-algebra will be the Power Set (the collection of all possible subsets). An event $A \in \mathcal{F}$ might be:

- "Was the first flip Heads?" $\rightarrow A = \{HH, HT\}$
- "Were both flips the same?" $\rightarrow A = \{HH, TT\}$
- "Did we get at least one Tail?" $\rightarrow A = \{HT, TH, TT\}$

Because we can ask these questions, their negations (complements) and combinations (unions) must also be in $\mathcal{F}$. For this finite Ω, $\mathcal{F}$ contains $2^4 = 16$ possible events.

Since the coin is fair, each elementary outcome has a probability of $1/4$. For any event $A \in \mathcal{F}$:

$$\mathbb{P}(A) = \frac{\text{Number of outcomes in } A}{4}$$

- $\mathbb{P}(\text{First flip is Heads}) = \mathbb{P}(\{HH, HT\}) = 2/4 = 1/2$
- $\mathbb{P}(\Omega) = \mathbb{P}(\{HH, HT, TH, TT\}) = 4/4 = 1$

Definition 2.5. A **random variable** X is a function $X : \Omega \rightarrow \mathbb{R}$ that assigns a real number to every outcome ω in the sample space. To be mathematically valid, it must be **measurable**. A random variable X is **measurable** with respect to a σ-algebra $\mathcal{F}$ if, given the information in $\mathcal{F}$, you can always determine the value of X.

Example 2.6

Suppose you flip a coin twice, so $\Omega = \{HH, HT, TH, TT\}$. We can define a random variable X as the number of heads:

- $X(HH) = 2$
- $X(HT) = 1$
- $X(TH) = 1$
- $X(TT) = 0$

Here, X is a measurable random variable with respect to $\mathcal{F}$ because we can calculate the probability of any question like "$X = 1$."

Definition 2.7. A **filtration** is an increasing sequence of σ-algebras

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F},$$

where $\mathcal{F}_n$ represents *all information available up to time n.*

In a stochastic process, information is revealed over time. At step n we know more than we did at step $n-1$.

Definition 2.8. The **natural filtration** of a process (X_n) is $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$: the information generated by the process up to time n.

Definition 2.9. A stochastic process $(X_n)_{n\geq 0}$ is called **adapted** to the filtration $(\mathcal{F}_n)$ if X_n is $\mathcal{F}_n$-measurable for every $n \geq 0$, i.e. the value of X_n is fully determined by the information available at time n. Equivalently: knowing $\mathcal{F}_n$, you can compute X_n exactly.

§3 Conditional Expectation

Definition 3.1. Let Y be a random variable and $\mathcal{G}$ be a σ-algebra. The **conditional expectation** of Y given $\mathcal{G}$, denoted $\mathbb{E}[Y \mid \mathcal{G}]$, is a random variable satisfying:

1. $\mathbb{E}[Y \mid \mathcal{G}]$ is $\mathcal{G}$-measurable (it only uses information available in $\mathcal{G}$).
2. For any event $A \in \mathcal{G}$, $\mathbb{E}[Y \cdot \mathbb{1}_A] = \mathbb{E}[\mathbb{E}[Y \mid \mathcal{G}] \cdot \mathbb{1}_A]$.

Definition 3.2. A random variable X is called **integrable** if $\mathbb{E}[|X|] < \infty$.

Theorem 3.3 (Properties of Conditional Expectation)

Let X, Y be integrable random variables and $\mathcal{G}, \mathcal{H}$ be σ-algebras.

1. **Linearity:** $\mathbb{E}[aX + bY \mid \mathcal{G}] = a\mathbb{E}[X \mid \mathcal{G}] + b\mathbb{E}[Y \mid \mathcal{G}]$ for any $a, b \in \mathbb{R}$.
2. **Stability:** If X is $\mathcal{G}$-measurable, then $\mathbb{E}[XY \mid \mathcal{G}] = X\mathbb{E}[Y \mid \mathcal{G}]$.
3. **Tower Property:** If $\mathcal{G} \subseteq \mathcal{H}$, then $\mathbb{E}[\mathbb{E}[Y \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[Y \mid \mathcal{G}]$.
4. **Independence:** If Y is independent of $\mathcal{G}$, then $\mathbb{E}[Y \mid \mathcal{G}] = \mathbb{E}[Y]$.
5. **Iterated Expectation:** $\mathbb{E}\big[\mathbb{E}[Y \mid \mathcal{G}]\big] = \mathbb{E}[Y]$.

Example 3.4

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables. Let $S_n = X_1 + \cdots + X_n$ be their sum. Compute the conditional expectation of a single component given the total sum:

$$\mathbb{E}[X_1 \mid S_n].$$

Solution. By symmetry, since the variables are identically distributed and their sum is fixed as S_n, the "best guess" for any individual X_i must be the same:

$$\mathbb{E}[X_1 \mid S_n] = \mathbb{E}[X_2 \mid S_n] = \cdots = \mathbb{E}[X_n \mid S_n].$$

Using the linearity of conditional expectation, we can sum these guesses:

$$\sum_{j=1}^{n} \mathbb{E}[X_j \mid S_n] = \mathbb{E}\left[\sum_{j=1}^{n} X_j \,\middle|\, S_n\right] = \mathbb{E}[S_n \mid S_n].$$

Since S_n is known, we have $\mathbb{E}[S_n \mid S_n] = S_n$. Thus:

$$n\mathbb{E}[X_1 \mid S_n] = S_n.$$

Dividing by n, we obtain:

$$\mathbb{E}[X_1 \mid S_n] = \frac{S_n}{n}.$$

If we only know the total sum of n identical variables, our best estimate for any one of them is simply the sample average. □

Problem 1 (5 points)**.** Let $X_1, X_2, \dots$ be i.i.d. random variables with mean μ. Let N be a positive-integer-valued random variable, independent of the sequence (X_i), with $\mathbb{E}[N] < \infty$. Let $S_N = \sum_{i=1}^{N} X_i$. Prove that $\mathbb{E}[S_N] = \mathbb{E}[N] \cdot \mu$.

Problem 2 (5 points)**.** You are on a gameshow and presented with two boxes designated A and B. You are given a choice between taking only box B or taking both boxes A and B. The player knows the following:

- Box A is transparent, or open, and always contains a visible $1,000.
- Box B is opaque and its content has already been set by the predictor:
  1. If the predictor has predicted that the player will take both boxes A and B, then box B contains nothing.
  2. If the predictor has predicted that the player will take only box B, then box B contains $1,000,000.
- The predictor predicts correctly 99% of the time.

You don't know what the predictor predicted or what box B contains while making the choice. You get decision paralysis and end up flipping a fair coin to decide whether to take just box B or both boxes. The predictor's 99% accuracy applies regardless of how the choice is made. Let Y be your total winnings. Find $\mathbb{E}[Y]$ with proof.

§4 Martingales

Definition 4.1. A stochastic process $(M_n)_{n\geq 0}$ is called a **martingale** with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ if the following three conditions hold:

1. M_n is $\mathcal{F}_n$-measurable for every $n \geq 0$.
2. $\mathbb{E}[|M_n|] < \infty$ for every $n \geq 0$.
3. $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$ for every $n \geq 0$.

If condition (3) is replaced by $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \leq M_n$, the process is called a **supermartingale** (expected to decrease – an unfavorable game). If replaced by $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \geq M_n$, it is a **submartingale** (expected to increase).

Essentially, a martingale is a process whose expected future value, given all present information, is equal to its current value. Martingales are *fair games*.

Definition 4.2. A non-negative integer-valued random variable T is a **stopping time** with respect to $(\mathcal{F}_n)$ if

$$\{T = n\} \in \mathcal{F}_n \quad \text{for all } n \geq 0.$$

Equivalently, $\{T \leq n\} \in \mathcal{F}_n$ for all n. The decision to stop at time n may only use information available at time n – no peeking into the future.

A stopping time is a strategy for deciding when to "act" (sell a stock, stop gambling, declare a winner) based only on what you've seen so far. The key feature is that you cannot use future information in the decision.

Example 4.3 (Stopping times vs. non-stopping times)

Let (S_n) be simple random walk.

- **Is a stopping time:** $T = \min\{n : S_n \geq 10\}$ (you decide to stop as soon as the walk hits 10). At each moment you only need to check the current value.
- **Is a stopping time:** $T = \min\{n : S_n \in \{-a, a\}\}$ (stop when the walk exits $(-a, a)$).
- **Is NOT a stopping time:** "The time at which S_n achieves its maximum over $n = 0, 1, \ldots, 100$." Knowing whether step n is the maximum requires knowing all future values.
- **Is NOT a stopping time:** "The last time the walk visits 0 before time 100." Again, you can't know this until time 100.

Proposition 4.4

If S and T are stopping times, then so are $S \wedge T := \min(S, T)$ and $S \vee T := \max(S, T)$.

§4.1 Collection of classics

Example 4.5

Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $\mathbb{P}(\xi_i = +1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$. Set $S_0 = x$ and $S_n = x + \xi_1 + \cdots + \xi_n$. Then (S_n) is a martingale with respect to its natural filtration: $\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[\xi_{n+1}] = S_n$.

Example 4.6

With S_n as above and $S_0 = 0$, the process $(S_n^2 - n)$ is also a martingale. Check: since $S_{n+1} = S_n + \xi_{n+1}$,

$$S_{n+1}^2 = S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2.$$

Taking conditional expectation: $\mathbb{E}[S_{n+1}^2 \mid \mathcal{F}_n] = S_n^2 + 0 + 1 = S_n^2 + 1$, so $\mathbb{E}[S_{n+1}^2 - (n+1) \mid \mathcal{F}_n] = S_n^2 - n$. This martingale encodes the variance of the random walk.

Example 4.7

For $\theta \neq 0$, let $\phi(\theta) = \mathbb{E}[e^{\theta\xi_1}]$ be the moment-generating function of a single step. Then

$$M_n = \frac{e^{\theta S_n}}{\phi(\theta)^n}$$

is a martingale. For the symmetric walk with $\mathbb{P}(\xi = \pm 1) = 1/2$, we have $\phi(\theta) = \cosh\theta$, so $M_n = e^{\theta S_n}/(\cosh\theta)^n$. When $p \neq 1/2$ and $\theta = \log(\frac{1-p}{p})$, this becomes $M_n = ((1-p)/p)^{S_n}$, which is exactly the martingale used to solve the asymmetric Gambler's Ruin.

Example 4.8

Let Y be any integrable random variable and $(\mathcal{F}_n)$ any filtration. Define

$$M_n = \mathbb{E}[Y \mid \mathcal{F}_n].$$

By the tower property, $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[Y \mid \mathcal{F}_n] = M_n$. So (M_n) is a martingale for any Y whatsoever. This is called the **Doob martingale**. Intuitively, as we learn more about the outcome, our running estimates of Y form a fair process.

Problem 3 (8 points)**.** Show that $(\alpha M_n + \beta N_n)$ is a martingale for any constants α, β.

Problem 4 (8 points)**.** Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}[X_i] = 0$. Let $S_n = \sum_{i=1}^n X_i$. Define $M_n = S_n/2^n$. Is (M_n) a martingale? Justify your answer by computing $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n]$ explicitly.

Problem 5 (8 points)**.** In an urn, we start with 1 red ball and 1 blue ball. Every step, we pick a ball, see its color, and put it back with an extra ball of the same color. Let R_n be the number of red balls after n draws. After n draws, the total number of balls is $n+2$. Prove that the proportion of red balls

$$M_n = \frac{R_n}{n+2}$$

is a martingale.

Hint: Compute $\mathbb{E}[R_{n+1} \mid \mathcal{F}_n]$ by conditioning on whether the drawn ball is red or blue.

Problem 6 (8 points)**.** Let $f : \mathbb{Z} \to \mathbb{R}$ be a function.

Prove that the process $(f(S_n))_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)$ if and only if

$$f(k) = \frac{f(k-1) + f(k+1)}{2} \quad \text{for all } k \in \mathbb{Z}.$$

§4.2 Optional Stopping Theorem

The Optional Stopping Theorem (OST) is the main computational tool in martingale theory. Essentially: you cannot beat a fair game, regardless of your stopping strategy.

Theorem 4.9 (Optional Stopping Theorem)

Let $(M_n)_{n\geq 0}$ be a martingale with respect to $(\mathcal{F}_n)$ and let T be a stopping time. Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ holds under any one of the following conditions:

(a) **(Bounded stopping time)** $T \leq N$ a.s. for some fixed integer N.

(b) **(Bounded process)** (M_n) is bounded, i.e. $|M_n| \leq C$ a.s. for some constant C, and $T < \infty$ a.s.

(c) **(Bounded increments + finite mean)** $\mathbb{E}[T] < \infty$ and $|M_{n+1} - M_n| \leq C$ a.s. for some constant C.

(d) **(Dominated convergence)** $T < \infty$ a.s. and there exists an integrable random variable Y with $|M_{T\wedge n}| \leq Y$ a.s. for all n.

For supermartingales the same conditions give $\mathbb{E}[M_T] \leq \mathbb{E}[M_0]$; for submartingales, $\mathbb{E}[M_T] \geq \mathbb{E}[M_0]$.

Here is some helpful advice on which condition you should try to show and when.

- **Use (a)** when the game must end by some fixed time N (e.g. a tournament with a round limit).
- **Use (b)** when the process is visibly bounded – e.g. a fortune that can never exceed N or drop below 0.
- **Use (c)** when you can show the game ends in finite expected time and the step sizes are bounded (most gambling/walk problems).
- **Use (d)** in advanced settings where the above fail but you can exhibit a dominating integrable random variable. Not necessary for any of the problems on this power round.

Problem 7 (8 points)**.** Let (S_n) be a simple symmetric random walk starting at $S_0 = 0$, and let $T = \min\{n : S_n = 1\}$. Then $T < \infty$ almost surely, and $S_T = 1$ always, yet $\mathbb{E}[S_0] = 0 \neq 1 = \mathbb{E}[S_T]$. Explain why this does not contradict OST.

§5 Worked examples

Throughout, $\mathbb{E}_x$ and $\mathbb{P}_x$ denote expectation and probability conditional on the process starting at $S_0 = x$.

Example 5.1 (Symmetric Gambler's Ruin)

A gambler starts with \$$x$ (with $0 < x < a$) and makes successive bets of \$1 on a fair coin flip. She stops when she either has \$$a$ or is bankrupt. What is her probability of reaching \$$a$?

Solution. Let S_n be her fortune and $T = \min\{n : S_n \in \{0, a\}\}$. Since $|S_n| \leq a$ always and $T < \infty$ a.s., condition (b) of OST applies, giving $\mathbb{E}_x[S_T] = S_0 = x$.

Since $S_T \in \{0, a\}$, letting $p = \mathbb{P}_x(S_T = a)$:

$$x = 0 \cdot (1-p) + a \cdot p \implies p = \frac{x}{a}.$$

The gambler's ruin probability is x/a: if she starts halfway to the goal, she has exactly a 50% chance. □

Example 5.2

In the same setup, what is the expected number of steps until the game ends?

Solution. Since $T \wedge n \le n$, condition (a) of OST applies to the stopped martingale $M_{T\wedge n}$, giving $\mathbb{E}_x[S_{T\wedge n}^2 - (T \wedge n)] = x^2$ for all n, i.e.

$$\mathbb{E}_x[S_{T\wedge n}^2] - \mathbb{E}_x[T \wedge n] = x^2.$$

As $n \to \infty$: since $|S_{T\wedge n}| \le a$, bounded convergence gives $\mathbb{E}_x[S_{T\wedge n}^2] \to \mathbb{E}_x[S_T^2]$. Since $T \wedge n \uparrow T$, monotone convergence gives $\mathbb{E}_x[T \wedge n] \to \mathbb{E}_x[T]$. Therefore $\mathbb{E}_x[S_T^2] - \mathbb{E}_x[T] = x^2$.

Since $S_T \in \{0, a\}$ and $\mathbb{P}_x(S_T = a) = x/a$, we have $\mathbb{E}_x[S_T^2] = a^2 \cdot (x/a) = ax$. Therefore

$$ax - \mathbb{E}_x[T] = x^2 \implies \mathbb{E}_x[T] = x(a-x).$$

For boundaries $\{-a, a\}$ with $S_0 = 0$ (shifting by a, $x = a$ with width $2a$), $\mathbb{E}_0[T] = a^2$. □

Example 5.3 (Asymmetric Gambler's Ruin)

Now suppose the gambler wins each bet with probability $p \ne 1/2$ and loses with probability $q = 1 - p$. Starting at \$$k$, what is the probability of reaching \$$N$ before going broke?

Solution. Let $\theta = q/p$. Then $M_n = \theta^{S_n}$ is a martingale:

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \theta^{S_n}\left(p \cdot \theta + q \cdot \theta^{-1}\right) = \theta^{S_n}(q + p) = M_n.$$

Let $T = \min\{n : S_n \in \{0, N\}\}$ be the exit time. Since $0 \le \theta^{S_n} \le \max(1, \theta^N)$, the process (M_n) is bounded, and $T < \infty$ a.s., so condition (b) of OST applies:

$$\theta^k = \mathbb{E}[M_T] = \mathbb{P}(S_T = N) \cdot \theta^N + \mathbb{P}(S_T = 0) \cdot 1.$$

Setting $q_k = \mathbb{P}(S_T = 0)$ and $1 - q_k = \mathbb{P}(S_T = N)$:

$$\theta^k = (1 - q_k)\theta^N + q_k,$$

so $q_k = \dfrac{\theta^k - \theta^N}{1 - \theta^N}$ and the probability of reaching N is

$$\mathbb{P}_k(S_T = N) = \frac{1 - \theta^k}{1 - \theta^N} = \frac{1 - (q/p)^k}{1 - (q/p)^N}.$$

□

Remark 5.4. As $p \to 1/2$ (i.e. $\theta \to 1$), applying L'Hôpital's rule:

$$\lim_{\theta \to 1} \frac{1 - \theta^k}{1 - \theta^N} = \lim_{\theta \to 1} \frac{-k\theta^{k-1}}{-N\theta^{N-1}} = \frac{k}{N},$$

consistent with Example 5.1.

Example 5.5

Suppose $f : \mathbb{Z} \to \mathbb{R}$ satisfies

$$f(k) = \frac{f(k-1) + f(k+1)}{2} \quad \text{for all } k \in \mathbb{Z}.$$

Prove that there exist real constants α, β such that

$$f(k) = \alpha k + \beta \quad \text{for all } k \in \mathbb{Z}.$$

Solution. The condition

$$f(k) = \frac{f(k-1) + f(k+1)}{2}$$

is equivalent to

$$f(k+1) - f(k) = f(k) - f(k-1).$$

Define

$$d_k = f(k) - f(k-1).$$

Then the equation says $d_{k+1} = d_k$ for all $k \in \mathbb{Z}$, so d_k is constant. Let this constant be α. Then

$$f(k) - f(k-1) = \alpha$$

for all $k \in \mathbb{Z}$.

Let $\beta = f(0)$. For $k > 0$, repeated use of the recurrence gives

$$f(k) = f(0) + k\alpha = \alpha k + \beta.$$

For $k < 0$, iterating the recurrence downward gives

$$f(k) = f(0) + k\alpha = \alpha k + \beta.$$

Therefore

$$f(k) = \alpha k + \beta \quad \text{for all } k \in \mathbb{Z}.$$

□

§6 More problems

Problem 8 (10 points)**.** There are n couples at therapy. For each couple, one of them is a mathematician and the other is a physicist. We randomly match the mathematicians to the n physicists (a random permutation). Any couple that is correctly matched is removed, and the process is repeated on the remaining pairs. What is the expected number of rounds until everyone is reunited?

Problem 9 (10 points)**.** $2N$ people stand in a circle. One person holds a coin. At each step, they flip it and pass it left (Tails) or right (Heads). The game ends when $2N - 1$ people have held the coin. What is the probability that the person diametrically opposite the starter is the **last** to touch the coin?

Problem 10 (10 points)**.** In the setup of the asymmetric gambler's ruin (Example 5.3), what is the expected number of steps until the game ends?

Hint: Define $d = p - q$. Note that $1 - d^2 = 4pq$. Show that $M_n = (S_n - nd)^2 - n(1 - d^2)$ is a martingale, and apply OST (you may assume $\mathbb{E}[T] < \infty$).