Algebra and Number Theory

PVMT 2025

All answers will be positive integers, all angles are less than 180°, and all diagrams are drawn to scale. All radical and logarithmic functions are assumed to give positive values. If a problem with a diagram has multiple configurations, refer to the configuration given in the diagram.

Problem 1 ([**2**])**.** Soham scored 25 points in a basketball game. He made only 2-point shots and 3-point shots. How many possible pairs of

(number of 2-point shots, number of 3-point shots)

could he have obtained?

Problem 2 ([**2**])**.** How many integer pairs (a, b) are there such that the three lines

$$y = x,\ y = -2x + a,\ y = 3x + b$$

all intersect at the same lattice point, and $|a|, |b| \leq 18$?

Problem 3 ([**3**])**.** Let N be the base 10 number $3! \cdot 4! \cdot 5!$. How many digits are in the base 2 conversion of N?

Problem 4 ([**3**])**.** Let the roots of $x^3 - 6x^2 + 14x + 3$ be x, y, and z. If the absolute value of

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}$$

can be written as m/n where m and n are integers and $\gcd(m, n) = 1$, what is $m + n$?

Problem 5 ([**3**])**.** Find the number of positive integer divisors of 720^2 that give a remainder of 1 when divided by 7.

Problem 6 ([**4**])**.** There are exactly three two-digit positive integers n that divide the sum of the nth powers of the digits of n. Find the sum of these three numbers.

Problem 7 ([**4**])**.** Let $f(n)$ be the greatest integer k such that 3^k divides $5^n - 2^n$. Find

$$\sum_{n=1}^{2025} f(n).$$

Problem 8 ([**4**])**.** Find all ordered pairs of integers for x, y between $1, 6$ inclusive, that satisfy

$$3^{x^y} = 5^{y^x} \mod 7.$$

Submit the sum of the y values of all these ordered pairs.

Problem 9 ([**5**])**.** Let $\{a_i\}$ be a sequence such that $a_i \in \{-1, 0, 1\}$ for all i and the product of the solutions for x to the equation

$$(\log_2 x - a_2)(\log_3 x - a_3)(\log_4 x - a_4)(\log_5 x - a_5)(\log_6 x - a_6) = a_1$$

is n, where n is the smallest integer value possible. For this value of n, determine the number of possible 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$.

Problem 10 ([**5**])**.** Consider the following function

$$S(n) = \sum_{k=1}^{n} (4k+1)^3.$$

What is the smallest n such that $S(n)$ is divisible by 225?

Problem 11. (Tiebreaker) The (p, q)-torus knot lying on the torus with equation $(r-2)^2 + z^2 = 1$ is given by the parameterization

$$x = r\cos(p\phi)$$

$$x = r\sin(p\phi)$$

$$x = -\sin(q\phi)$$

where $r = \cos(q\phi) + 2$ and $0 < \phi < 2\pi$ (in spherical coordinates).

The Jones polynomial of a right-handed torus knot is given by

$$t^{(p-1)(q-1)/2} \frac{1 - t^{p+1} - t^{q+1} + t^{p+q}}{1 - t^2}.$$

Estimate as a decimal the nonzero real root of the Jones polynomial of the right-handed $(20, 25)$-torus knot.