PVMT 2024: Team Round Solutions

Problem 1

Evaluate

$$(\frac{\sqrt{2}+2}{1+\sqrt{2}})^2(1^{\log_{12}21})$$

Proposed by Milo Stammers

Solution

The fraction can be reduced to $\sqrt{2}$, and 1 to the power of anything is simply 1. Thus is comes out to $(\sqrt{2})^2(1) = \boxed{2}$.

Problem 2

A random permutation of the numbers 2,0,2,4 is chosen and assigned, in the order they are permuted, to the variables P, V, M, and T, respectively. The probability that the value of the expression $\frac{\frac{P}{V}}{\frac{M}{T}}$ is defined is $\frac{p}{q}$. What is $p+q$?

Proposed by Anna Zhou

Solution

For the expression to be defined, none of the denominators can equal 0. P is the only variable that can equal 0, at which point the values of the other variables don't matter. The probability that P equals 0 is $\frac{1}{4}$ and the answer is $\boxed{5}$.

Problem 3

Let PQR be a 30-60-90 triangle with hypotenuse $PQ = 2$ and $\angle Q = 60^\circ$. Let circle Ω be tangent to PR and RQ with a diameter AB where A and B are on segment PQ and A is closer to P than Q. $BQ = \frac{a\sqrt{b}-c}{d}$ for positive integers a,b,c,d with b squarefree. What is $a+b+c+d$?

Proposed by Alyssa Yu

Solution

We have $PR = 1$. Let O be the center of Ω and C and D be the intersections of Ω with PR and QR, respectively. Thus $CODR$ must be a square and let its side length be x. We have

$$QR = QD + DR = \frac{1}{\sqrt{3}}x + x = 1$$

so $x = \frac{3-\sqrt{3}}{2}$. Looking at $\triangle ODQ$, $WB = QO - BO = x\frac{2}{\sqrt{3}} - x = \frac{3\sqrt{3}-5}{2}$, and our answer is $\boxed{13}$.

Problem 4

Let a, b, c, d, e be distinct integers from -2 to 2. If $a + b^2 + c^3 + d^4 + e^5 = 10$, what is the value of $a + 2b + c^2 + 2d + e$?

Proposed by Ricky Sun

Solution

We can deduce that $b = -1$, $c = -2$, and $d = 2$ while a and e are 0 and 1. However, both configurations lead to the same answer. This gives $a + 2b + c^2 + 2d + e = \boxed{7}$.

Problem 5

There is a class of students at Poolesville such that exactly 400 different pairs of those students are friends and it is possible to divide the students into 3 groups such that each group contains no friendships. What is the minimum possible size of this class?

Proposed by Ricky Sun

Solution

We will label the 3 groups of size a, b and c and have each student be friends with everyone that isn't in the same group as them. The resulting maximum amount of friendships is $ab + bc + ac$ which is maximized when $a = b = c$ resulting in $3a^2$ friendships. Solving $3a^2 = 400$ for $3a$ and rounding up yields $\boxed{35}$.

Problem 6

How many ways are there to arrange the string "122345" such that no number goes into its original position (neither of the 2's can go into positions 2 or 3)?

Proposed by Alex Wang, Solution by Anna Zhou

Note: This problem was scratched from the competition for being a repeat problem from the previous year.

Solution

We choose where we put the 2's first, and multiply by $\binom{4}{2} = 6$ at the end. Without loss of generality, we assume the 2's go into positions 1 and 4 (note that 3 is in position 4). Now 1 and 3 are free to go wherever they like, so we do casework on the positions of 4 and 5.

Case 1: 4 and 5 go into positions 2 and 3, where the 2's were originally. 1 and 3 must go into positions 5 and 6. We have $2 * 2 = 4$ for this case.

Case 2: Either 4 or 5 go into positions 2 or 3. The other switches its position to 5 or 6. 1 and 3 take the remaining two positions. We have $2 * 2 * 2$ for this case.

Case 3: 4 and 5 stay in positions 5 and 6 and switch places. 1 and 3 go in positions 2 and 3. We have 2 for this case. The sum is 14, and we multiply by 6 from earlier to get $\boxed{84}$.

Problem 7

Let $n = 100! + 101$. Find the sum of the last 20 digits of n^7.

Proposed by Logan Van Pelt, Solution by Milo Stammers

Solution

Taking $(100! + 101)^7 \mod 10^{20}$, if we apply binomial theorem, all but the 101^7 term has a factor of at least 10^{20} in 100!. Thus we wish to find the last 20 digits of 101^7. Note that $101^2 = 1,02,01, 101^3 = 1,03,03,01$, and so on with the numbers being that of the binomial distribution. Thus $101^7 = 1,07,21,35,35,21,07,01$ and the sum of the digits is $\boxed{38}$.

Problem 8

In a circular room with radius 1, the entire wall is covered with mirrors. A person is standing $\frac{9}{10}$ units from the center of the room and shines a laser such that the beam forms a regular hexagonal path. Let the distance the beam travels from the person to where it first bounces off a wall be d, the absolute difference between the two possible values of d is $\frac{\sqrt{a}}{b}$. What is ab?

Proposed by Milo Stammers

Solution

To form a regular hexagon, the beam must traverse an arc of $\frac{360}{6} = 60$ degrees between each bounce. On the first and last bounces, the beam will start/end at the person. If we draw a triangle at with vertices at the center of the room, O, and the locations of the first and last bounces, A, B. OAB will be isosceles with vertex angle of 60, thus is equilateral with length 1. Let point P be the location of the person, thus P lies on AB with $OP = \frac{4}{5}$. Let $AP = m, PB = 1 - m$, apply Stewart's Theorem to get

$$m(1-m) + \frac{81}{100} = 1 - m + m$$

$$0 = m^2 - m + \frac{19}{100}$$

$$m = \frac{1 \pm \sqrt{\frac{6}{25}}}{2}$$

Thus the difference between the solutions is $\sqrt{\frac{6}{25}} = \frac{\sqrt{6}}{5}$, giving $\boxed{30}$.

Problem 9

At the center of regular hexagon $ABCDEF$ lies equilateral triangle PQR such that $BCQP$, $DERQ$, and $FAPR$ are rectangles. A triangle is randomly chosen in the hexagon, the probability that the triangle does not intersect any of the rectangles is $\frac{m}{n}$. What is $m + n$?

Proposed by Milo Stammers

Solution

The rectangles will split the hexagon into four completely separated regions, three $30-30-120$ triangles and the equilateral triangle in the center. Let the side length of the hexagon equal 1. Then the small triangles have lengths $\frac{1}{\sqrt{3}}$, and thus an area of $\frac{1}{4\sqrt{3}}$. Thus they make up a fraction $\frac{1}{18}$ of the entire hexagon. The side length of the equilateral triangle is equal to the side length of the rectangles and thus the hexagon, so has area $\frac{\sqrt{3}}{4}$, or fraction $\frac{1}{6}$. All vertices must lie in the same region to not intersect the rectangles, so the probability is $3\frac{1}{18^3}+\frac{1}{6^3}=\frac{5}{972}$, giving us the answer $\boxed{977}$.

Problem 10

A robot is standing at the origin of a Cartesian plane. It can only move right or up by one unit, and the goal is to reach the point $(7,7)$. However, there are some blocked points in the plane where the robot cannot step on. The blocked points are on $(1,2)$, and $(5,6)$. How many different paths can the robot take to reach its goal at point $(7,7)$?

Proposed by Soham Kyada, Solution by Anna Zhou

Solution

We do complementary counting. There are $\binom{14}{7}=11*13*24=3432$ ways for the robot to get to $(7,7)$ if there are no blocked points. There are $\binom{3}{1}*\binom{11}{5}=3*42*11=1386$ paths that go through $(1,2)$, and the same for $(5,6)$. However, we overcount the paths that pass through both blocked points: $\binom{3}{1}*\binom{8}{4}*\binom{3}{1}=9*70=630$. The answer is $3432-1386*2+630=\boxed{1290}$.

Problem 11

Consider the following curve,

$$(x+y)((x+y)^2-x-y-3)=2-4xy$$

The closest this curve gets to the origin squared is $\frac{a}{b}$. What is $a+b$?

Proposed by Milo Stammers

Solution

Inside the left hand side, x^2+y^2+2xy can be factored, then expanded into

$$(x+y)^3-(x+y)^2-3(x+y)=2-4xy$$

We are here motivated to move the $(x+y)^2$ to the right hand side because when subtracting $4xy$, we will be left with $(x-y)^2$. We let $u=x+y$, $v=x-y$

$$u^3-3u-2=v^2$$

The distance to the origin squared is equal to $x^2+y^2=\frac{1}{2}(u^2+v^2)$, so we wish to minimize u^2+v^2. The left can be factored into $(u+1)^2(u-2)=v^2$. For $u\in(-1,2)$, the left is negative so there is no real solution for v. u^2 is thus minimized at -1 in which $v=0$. Because v^2 cannot be less than 0 and u^2 less than 1, the minimum distance equals $\sqrt{\frac{(-1)^2+0^2}{2}} = \frac{1}{\sqrt{2}}$ so the answer is $\boxed{3}$.

The graph here is that of an elliptic curve, and through the substitution, it is rotated 45°. These curves have a number of interesting properties and applications.

Problem 12

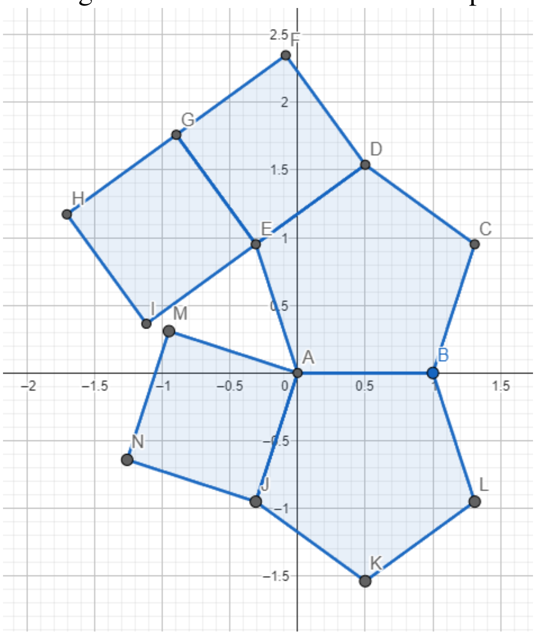
Consider 5 regular polygons from the set of triangles, squares, and pentagons all with equal side length. These polygons are arranged on a plane without any overlapping areas, such that each newly placed polygon shares at least one edge with an already placed polygon. In the configuration that contains the minimum possible distance between 2 vertices, how many total vertices are there? Overlapping vertices are counted as a single vertex.

Proposed by Ricky Sun

Solution

Consider some cases: First case, all polygons share a vertex. In this case, getting the interior angle as close to 360 is optimal. Since the interior angles are $60, 90$ and 108, we can add their multiples to get as close as possible. This can be done using 4 triangles and a pentagon, summing up to an interior angle of 348 and an isosceles triangle of 12 degrees. The smallest distance between vertices would be between the 12 degree angle.

Second case, there are 2 vertices connected by a single edge such that any polygon has at least one of those two vertices. There will be one polygon that has both. In this case, the two interior angles summing to 600 with relatively equal distribution will almost form an equilateral triangle, optimizing the distance. If it were to go beyond 600, then two edges would overlap. The middle polygon will use its interior angle twice. We can do similar calculations to find that using 2 pentagons and 3 squares with a pentagon in the middle, the interior angles sum to 306 and 288. See the picture below.



Third case is when there are 3 vertices that span all polygons connected by 2 edges. In this case we try to create a rhombus. 5 pentagons overlaps the 2 vertices so the closest is appending 4 pentagons and a square, creating an isosceles triangle with 18 degrees. This is worse than the first case so we disregard it.

Last case is when there are 4 or more vertices that span all polygons connected by 3 edges. Since we only have 5 polygons, it can be easily seen that this will not provide 2 close vertices.

Comparing the optimal length for cases one and two, we can use isosceles triangles to estimate the second case to a form similar to the first. $360-288-(306-180)/2$ giving us 9 degrees in the second case. Thus the second case appears to be smaller. To be extra sure of our answer, we can compare the error from the estimate of the second case using the law of cosines, but it's too negligible compared to the angle. Thus, counting the second case, there are $\boxed{14}$ total vertices.

Problem 13

The value of

$$\sum_{n=1}^{\infty} \frac{n\sin(n)}{2^n}$$

can be represented as $\frac{n\sin(1)}{([2-\cos(1)]^2+\sin^2(1))^2}$. What is n?

Proposed by Milo Stammers

Solution

Write $\sin(n)$ as the imaginary component of e^{in}. We can now write the series as

$$\sum_{n=1}^{\infty} n(\frac{e^i}{2})^n$$

Let $r=\frac{e^i}{2}$ for clarity. We now wish to evaluate the series

$$S=\sum_{n=1}^{\infty} nr^n = 1*r+2*r^2+3*r^3+...$$

Therefore,

$$rS = 0*r+1*r^2+2*r^3+...$$

Subtracting these expressions

$$S(1-r) = r+r^2+r^3+... = \frac{r}{1-r}$$

$$S = \frac{r}{(1-r)^2}$$

Putting in $r=\frac{e^i}{2}=\frac{1}{2}(\cos(1)+i\sin(1))$ and multiplying by 4 on the top and bottom, we reduce the expression to

$$\frac{2(\cos(1)+i\sin(1))}{(2-\cos(1)-i\sin(1))^2}$$

If we multiply the numerator and denominator by $([2-\cos(1)]+i\sin(1))^2$, we make the denominator real and in the form of our final answer. We now wish to find the imaginary part of

$$2(\cos(1)+i\sin(1))([2-\cos(1)]+i\sin(1))^2$$

Multiplying out, this becomes

$$2(2\cos(1)(2-\cos(1))(\sin(1))+\sin(1)(4+\cos^2(1)-4\cos(1)-\sin^2(1)))=$$

$$8\sin(1)\cos(1)-4\sin(1)\cos^2(1)+8\sin(1)+2\sin(1)\cos^2(1)-8\sin(1)\cos(1)-2\sin^3(1)=$$

$$8\sin(1)-2\cos^2(1)\sin(1)-2\sin^3(1)=$$

$$8\sin(1)-2\sin(1)+2\sin^3(1)-2\sin^3(1)=6\sin(1)$$

By the Pythagorean Identity. Thus our answer is $\boxed{6}$.

Problem 14

In triangle ABC, let H be the orthocenter and O be the circumcenter. The circumradius of ABC is 7. Given that $AO=AH$ and OH is 2, the distance between A and the minor arc BC can be expressed as $a\sqrt{b}$ for integers a and b, where b is squarefree. What is $a\cdot b$?

Proposed by Sumedh Vangara

Solution

Let M be the midpoint of minor arc BC. Since AH and AO are isogonal conjugates, $AHOM$ is a parallelogram. Thus, using parallelogram law, we get that the answer is

$$\sqrt{4\cdot 49-4}=8\sqrt{3}$$

Thus $a\cdot b=\boxed{24}$

Problem 15

Sumedh is standing at the point $(\frac{1}{3},\frac{1}{4})$ in a room with walls on the unit circle. Sumedh walks around until he hits the wall. If he hits the wall at angle θ, then he scores $\cos(2\theta)$ points. Sumedh's expected score is $\frac{p}{q}$, what is $p+q$?

Proposed by Milo Stammers

Solution

Say Sumedh starts at some point (x,y). Consider some small circle centered at (x,y) completely contained in the unit circle. When Sumedh is walking, he is equally likely to reach any point on that circle first, so the expected score at that point is the average of the expected scores on the boundary of any circle centered at that point. This is called the mean value property, and I will refer to it as such for the remainder of the solution. Let $f(x,y)$ be the expected score for some point (x,y).

On the unit circle, $f(x,y)=\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)=x^2-y^2$. Let $g(x,y)$ be another function with the mean value property that also equals x^2-y^2 on the unit circle. Then

$d(x,y) = f(x,y) - g(x,y)$ must also have the mean value property and equals 0 on the unit circle. Note also that because a point is the average of those around it, there are no local maximums of minimums, otherwise the point would be larger than all those at a certain radius, the would thus be greater than the average (this is the Maximum Principle). Clearly, because $d(x,y)$ has a constant boundary and no max or mins, it must be identically 0 over the whole circle, thus $g(x,y) = f(x,y)$.

This is very powerful because it tells us that if we can find a function with the given boundary and mean value property, it is unique and thus our answer. By symmetry, if Sumedh stands on the line $x = y$ or $x = -y$, then his expected score is 0. Thus $x^2 - y^2$ divides $f(x,y)$, and the quotient must equal 1 on the unit circle. Finally, we are inclined to guess $x^2 - y^2$ is $f(x,y)$. To prove $x^2 - y^2$ has the mean value property, take $(x + r\cos(\theta))^2 - (y + r\sin(\theta))^2 = (x^2 - y^2) + 2r\cos(\theta) - 2r\sin(\theta) + r^2\cos^2(\theta) - r^2\sin^2(\theta)$. Averaging over 0, we find that it does in fact satisfy the mean value property. Thus our answer is $\frac{1}{9} - \frac{1}{16} = \frac{7}{144}$, giving us $\boxed{151}$.

We could more easily have gotten the solution with calculus noting that $x^2 - y^2$ is a harmonic function. In regular cases, harmonic functions and functions with mean value property are equivalent. Given an arbitrary boundary condition, we could find a unique harmonic function to describe the expected value over the center of the disk, but the general solution is difficult and is known as the Dirichlet Problem.