Problem 1

Square A has vertices $(0,0)$, $(6,0)$, $(0,6)$, and $(6,6)$. Square B has half the area of A and each vertex of B has integer coordinates. If B is fully contained in A, what is the product of the sums of the x and y coordinates of each vertex of B?

Proposed by Ricky Sun

Solution

B must have side length $3\sqrt{2}$. The only way for this to have integer coordinates is if B has vertices $(0,3)$, $(3,0)$, $(3,6)$, and $(6,3)$. Multiply the sums all to get $\boxed{729}$.

Problem 2

Right triangle ABC with right angle at B has perimeter $4+2\sqrt{6}$. Median BD has length 2. What is the area of ABC?

Proposed by Anna Zhou

Solution

Let $AB = a$ and $BC = b$. Note that* $BD = AD = CD$. It follows that the hypotenuse of ABC is 4. Then $a+b = 2\sqrt{6}$ and $a^2+b^2 = 16$. We have $2ab = (a+b)^2-(a^2+b^2) = 24-16 = 8$, and the area of ABC is $\frac{ab}{2} = \boxed{2}$.

*To see this, one can draw a line through D parallel to BC intersecting AB at E. Since AE=BE and $\angle AED = \angle BED = 90^\circ$, triangle ADB is isosceles.

Problem 3

In rectangle $ABCD$, let $AB = 3$, $BC = 4$. Let M be the midpoint of BC and N the midpoint of CD. The area of the triangle enclosed by AM, BN, and AB can be written as m/n where m and n are relatively prime positive integers. Find m^2+n^2.

Proposed by Milo Stammers

Solution

Let X be the intersection of AM and BN. Now scale the entire rectangle down by a factor of $\frac{3}{4}$ parallel to BC. All areas have been scaled down by that same factor. Notice that $ABCD$ now becomes a square, and so AM and BC intersect at a right angle. This means that triangle ABX is similar to AMB by angle-angle. The area of AMB is $\frac{1}{2}(3)\frac{3}{2} = \frac{9}{4}$. To find the side ratio of these triangles, $AM = \frac{3\sqrt{5}}{2}$ by Pythagorean Theorem, corresponding to $AB = 3$. Thus the area of $ABX = (\frac{3(2)}{3\sqrt{5}})^2\frac{9}{4} = \frac{9}{5}$. Converting back to our original rectangle, we must multiply by $\frac{4}{3}$ to get $\frac{12}{5}$. Thus $m^2+n^2 = \boxed{169}$.

Problem 4

Define points B, A, C at $(0,0),(1,0),(2,0)$. Let circle O be a circle with radius 1 centered at A. Let P be on O and let the tangent at P to circle O be line L. Let the closest point on line L to point C be point X. Define point Y such that Y, X, C are collinear in that order and $XC \cdot YC = 2$. Suppose Y has a x-coordinate of 1.5. What is $10 \cdot YC$?

Proposed by Alex Wang, Solution by Alyssa Yu

Solution

Let D be $XP \cap BC$ and Z be $(1.5,0)$. Also denote BD to be k. Then

$$DPA \sim DXC \sim YZC.$$

Therefore by length ratios we have $XC = \frac{k+2}{k+1}$ and $YC = \frac{k+1}{2}$, and $XC \cdot YC = \frac{k+2}{2} = 2 \implies k = 2$. Therefore our answer is $10 \cdot YC = \frac{10(k+1)}{2} = \boxed{15}$.

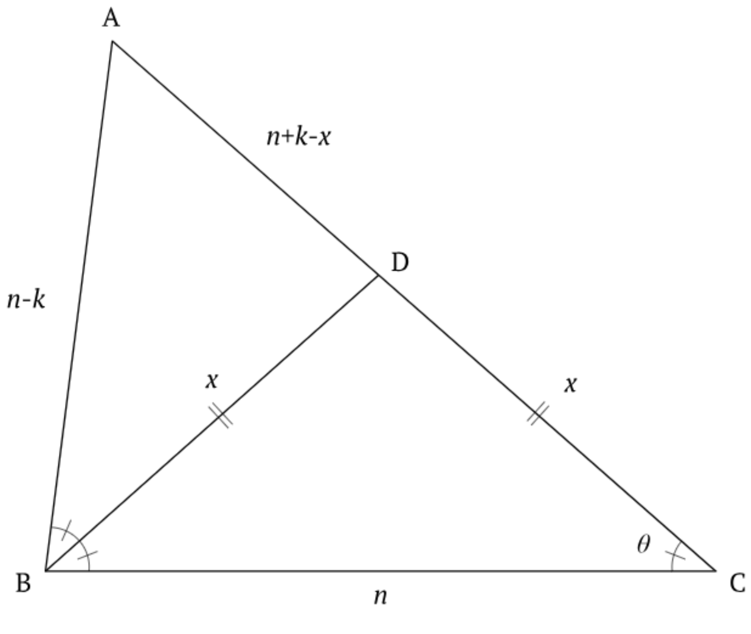
Note that the locus of Y is a cardioid under inversion which is a parabola.

Problem 5

Triangle ABC has integer side lengths that form an arithmetic progression. The largest angle is twice the smallest angle. Given that one of the sides has length 2024, what is the perimeter of ABC?

Proposed by Anna Zhou

Solution



We don't know which side is 2024 and it would be messy to consider three separate cases. It's also a rather big number. We ignore it for now and set $AB = n-k$, $BC = n$, and $AC = n+k$. We have $\angle ABC = 2\angle ACB$.

We draw the angle bisector BD to create an isosceles triangle BDC. Let $x = BD = DC$. We have $ADB\ ABC$. From this we can write two equations:
$\frac{n-k}{x} = \frac{n+k}{n} \Longrightarrow n^2 + kn - nx = nx - kx$
$\frac{n-k}{n+k-x} = \frac{n}{x} \Longrightarrow n^2 - kn = nx + kx$

Note that the second equation can also be obtained from the Angle Bisector Theorem on ABC). We hope to get rid of k and establish a direct relationship between n and x. Add the above two equations to get $2n^2 - nx = 2nx \Longrightarrow x = \frac{3}{2}n$. Plugging this into the second equation gives $n = 5k$.
The sides of ABC are in ratio 4:5:6. Since 2024 is only divisible by 4, we have that the perimeter is $2024 * \frac{15}{4} = \boxed{7590}$.

Solution 2: Trigonometry
Refer to the same diagram. Let $\angle ACB = \theta$ and $\angle ABC = 2\theta$. Law of Sines yields $\frac{2\sin 2\theta}{n+k} = \frac{\sin\theta}{n-k}$. Substituting $\sin 2\theta = 2\sin\theta\cos\theta$, we get $\frac{2\cos\theta}{n+k} = \frac{1}{n-k}$.
Law of Cosines gives $\cos\theta = \frac{n^2+(n+k)^2-(n-k)^2}{2n(n+k)} = \frac{n+4k}{2(n+k)}$. Plugging this back in yields $5k = n$. Proceed as in Solution 1.

Problem 6

In triangle ABC, $AB = 2$, and BD is the altitude from B to AC. Suppose A, B, C are chosen such that $BD = \frac{AC}{2}$. The minimum value of BC can be written as $a\sqrt{b} - c$. What is $100a \cdot b \cdot c$?

Proposed by Alex Wang, Solution by Milo Stammers

Solution

Let $\angle BAC = x$, then $\triangle BAD$ is right, so $BD = 2\sin(x)$. This implies $AC = 4\sin(x)$, which gives us the distance to C from A for a given angle. Considering all possible angles, this would give us the graph $r = 4\sin(\theta)$ in polar coordinates, with our axis being line AB. This graph will be a circle tangent to AB and A with diameter 4 (this may be scene by taking $x = r\cos(\theta), y = r\sin(\theta)$ and getting the equation of the circle). The minimum BC is the closest point on the circle to B. Taking O to be the center of the circle, $\triangle BAO$ has right angle at A, so $BO = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. Therefore the closest distance to the circle is $2\sqrt{2} - 2$, and this gives us $\boxed{800}$.

Problem 7

Let A and B be $(1,5)$ and $(3,1)$, respectively. Let P be a point that is distance 4 away from $(37,15)$. The sum of the coordinates of P when $PA^2 + PB^2$ is maximized can be represented as m/n where m and n are relatively prime positive integers, find $m+n \mod 1000$.

Proposed by Alyssa Yu

Solution

Let O be $(37,37)$. Translate the entire graph so that M, the midpoint of AB, is at $(0,0)$. From now on, $A = (-1,2)$, $B = (1,-2)$, $M = (0,0)$, and $O = (35,34)$. By the Parallelogram Law, $2(AP^2 + BP^2) = AB^2 + (2MP)^2$. Therefore we want to maximize MP. To maximize MP, we must have it go through $(35,12)$. Thus $P = (35 + 4 \cdot \frac{35}{37}, 12 + 4 \cdot \frac{12}{37})$. Since we shifted 2 down and 3 to the left, the value of P pertaining to the original problem is $(37 + 4 \cdot \frac{35}{37} + 2, 15 + 4 \cdot \frac{12}{37} + 3)$. The sum is $52 + 4 \cdot \frac{47}{37} = \frac{2112}{37}$. The answer is $\boxed{149}$.

Problem 8

Square $ABCD$ has side length 7. Draw a quarter-circle centered at B going through A and C. Let E be a moving point on this quarter circle. Let F be the point on segment BE such that $BF = 3$. The minimum value of $CF + DE$ can be written as $\sqrt{x}$. What is x?

Proposed by Alex Wang

Solution

Let P be the point on segment BC such that $BP = 3$. Then $DE + CF = DE + EP$ by reflecting about the angle bisector of $\angle CBE$. The minimum value is the length of the straight line from D to P which is $DP = \sqrt{65}$. Thus the answer is $\boxed{65}$.

Problem 9

Let $\triangle ABC$ have internal angle bisector AD and external angle bisector AE, where B, D, C, E are collinear in that order. Given that $CE = 8$ and $DE = 10$, the length BE can be written as m/n where m and n are relatively prime positive integers. What is $m + n$?

Proposed by Alyssa Yu

Solution

In the following we prove a result for external angle bisectors.

Lemma 1. $\frac{DC}{BD} = \frac{CE}{BE}$.

Proof. Let P be the point on segment AB such that $PC \parallel AE$. Then by angle chasing we see that $\angle APC = \angle ACP$ so $AP = AC$. Because $BPC \sim BAE$, we have

$$\frac{PA}{BA} = \frac{CE}{BE} \implies \frac{CA}{BA} = \frac{CE}{BE}.$$

By the internal angle bisector theorem we have $\frac{CA}{BA} = \frac{CD}{BD}$ so we have

$$\frac{DC}{BD} = \frac{CE}{BE}.$$

$\square$

From $\frac{DC}{BD} = \frac{CE}{BE}$ we have $\frac{DE-EC}{BE-DE} = \frac{CE}{BE}$. Plugging in $DE = 10$ and $CE = 8$ and cross multiplying, we have $BE = \frac{40}{3}$ so the answer is $\boxed{43}$.

Problem 10

Let ABC be a triangle where $AB = 5$, $BC = 8$, and $AC = 7$. Let I_B and I_C be the B-excenter and C-excenter of ABC (the excenter is the center of the circle tangent to one side and the extensions of the other two sides. For example the A-excenter is the center of the circle tangent to the extensions of AB and AC and segment BC; the B and C-excenters are defined similarly). The value $I_BI_C \cdot BC + BI_C \cdot CI_B$ can be written as $\frac{20\sqrt{x}}{3}$ where x is a positive integer. Find x.

Proposed by Alyssa Yu

Solution

Let C_1 and B_1 be the feet of the altitudes from I_C and I_B to BC, respectively. By the properties of the excircles, we have $C_1C + C_2C = 5+7+8 = 20$ and $C_1C = C_2C$ so $C_1C = C_2C = 10$. Thus $C_1B = 2$. Additionally let A' the foot from the incenter of ABC (which we will call I) to BC. The semiperimeter s is equal to 10, so we have $BA' = 5+8-s = 3$ and $CA' = 7+8-s = 5$. Moreover, by Heron's formula we can calculate $[ABC] = 10\sqrt{3}$ so the radius IA is $K/s = \sqrt{3}$. We have $I_CC_1C \sim IA'C$ so $I_CC_1 = IA' \cdot \frac{C_1C}{A'C} = 2\sqrt{3}$. Doing a similar thing with the B-excenter gives $I_BB_1 = 10/\sqrt{3}$.

Thus, by Pythagorean theorem, $I_CC = \sqrt{I_CC_1^2 + C_1C^2} = 20/\sqrt{3}$ and $I_BB = \sqrt{I_BB_1^2 + B_1B^2} = \sqrt{112}$. Notice that I_CBCI_B is cyclic because $I_CCI_B = I_CBI_B = 90°$. Therefore by Ptolemy's theorem $I_BI_C \cdot BC + BI_C + CI_B = I_CC \cdot I = 20\sqrt{336}/3$ and the answer is $\boxed{336}$.

Problem 11 (Tiebreaker)

A cyclic pentagon $ABCDE$ inscribed in a circle with center O can be divided into 3 kites $OABC$, $OCDM$, and $OMEA$ where M is the midpoint of DE. Estimate the ratio between the area of pentagon $ABCDE$ and the area of circle O. Write your answer in the form $0.abcdef$.

Proposed by Ricky Sun

Solution

The only way for the kites to fit is if $OCDM$, and $OMEA$ are congruent. Then through some tedious calculations we find the exact value of the ratio is $\frac{1+\sqrt{3}+\sqrt{2+\sqrt{3}}}{2\pi}$ which is approximately $\boxed{0.742283}$