Problem 1

Anna picks an edge and a face of a cube at random. What is the probability that they share at least one vertex, rounded to the nearest percent?

Proposed by Milo Stammers and Anna Zhou

Solution

For any edge, there are 4 out of 6 faces that share a vertex with it. The answer is $\frac{2}{3} \approx \boxed{67}\%$.

Problem 2

Soham starts at $(0,0)$ and randomly chooses with equal probability of $\frac{1}{5}$ between moving 1 unit left, 2 units down, 3 units right, 4 units up, and stopping. Soham keeps doing this until he stops. What is the expected value of the sum of the x and y coordinates that Soham stops at?

Proposed by Ricky Sun

Solution

Going up and right add to the expected value while going left and down subtract. Thus each move without stopping yields an expected gain per move of $(-1-2+3+4)/4 = 1$. Since each move has a $1/5$ chance of stopping, the expected moves before stopping is 4. Multiplying yields $\boxed{4}$.

Problem 3

Take a set of positive integers $S = \{a_1, a_2, ..., a_7\}$. Let A, B be disjoint subsets of S. Find the largest n such that all positive integers from 1 to n can be expressed as the sum of the elements in A minus the sum of the elements in B with some fixed choice of S.

Proposed by Milo Stammers

Solution

Instead of 7 elements, consider some smaller values first. For 1, obvious the maximum $n = 1$ with $S = \{1\}$. For 2, if we take the set $S = \{1,3\}$, we can get 1 with $\{1\}, \{\}$, 2 with $\{3\}, \{1\}$, and so on up to $n = 4$. For 3, we add the element 9 to S which now lets us also attain $5-13$, any larger would make 5 impossible to attain, and any smaller would make 13 impossible. Continuing this pattern, we add successive powers of three, and the best we can do is $1+3+3^2+...+3^6 = \frac{3^7-1}{2} = \boxed{1093}$.

Alternatively, we could also see this as having three options for each element, nothing, add it, or subtract it, so 3^7 possible elements at most. However, taking nothing leaves a sum of 0, and switching adds/subs would negate the value, so at most we can have $\frac{3^7-1}{2}$ distinct positive values.

Problem 4

Kabosu farms 1 dogecoin per day, Monday through Thursday. On Friday Kabosu either triples her dogecoin with probability $\frac{1}{4}$ or sells all of her dogecoin with probability $\frac{3}{4}$. She does nothing on the weekend. Kabosu repeats this until the end of time. After a long, long time, what is the expected number of dogecoin she has?

Proposed by Anna Zhou

Note: The original solution incorrectly assumed that the expected number of dogecoin is the same across the whole week after a sufficiently long time. Thanks to Roy Brauwerman for pointing out the mistake during the contest!

Solution

Suppose it is Monday. Let the expected number of Kabosu's dogecoin at the end of the day be M. The expected number of dogecoin at the end of Thursday will be $M+3$. The expected number of dogecoin at the end of Friday will be $\frac{3}{4}*(M+3)$. It stays the same over the weekend, so the expected number of dogecoin at the end of the following Monday will be $\frac{3}{4}*(M+3)=1$. This should be equal to M. Solving, we have $M=13$. That means the expected number of dogecoin is 13, 14, 15, 16, 12, 12, 12 for Monday through Sunday, respectively. Since there is a $\frac{1}{7}$ probability of being a particular day, the expected number of dogecoin after a "long, long time" is simply the average of each day's expected value, so the correct answer is $13\frac{3}{7}$.

For the official contest, we accepted any integer answer between 12 and 16 inclusive. Thanks again to Roy Brauwerman for correcting our mistake!

Problem 5

The PVMT team visits a park with seven falcon nests, numbered 1-7. After a while, everyone says they can see nests 1-5 but not nests 6 or 7. The only way you can't see a nest is if another nest is directly blocking it. No three nests are collinear and everything is on the same plane. No human is at a nest and no two humans occupy the same spot. At most how many members are on the PVMT team?

Proposed by Anna Zhou, Solution by Milo Stammers

Solution

Let A be the set of lines through nests $1-5$ and through 6, and B be lines through $1-5$ and 7. For $6,7$ to not be visible, a person must be sitting at an intersection of lines in A and B, on the opposite side of the lines to the nests. Assuming no two lines are parallel, then for any two nests a,b of $1-5$, there is at most one point such that a,b block $6,7$. This leaves $\binom{5}{2}=\boxed{10}$ places for people to stand.

This can be seen because there are two lines going through each a and b creating four intersections. Two of these are at nests $6,7$, and assume that $a6$ and $b7$ lines intersect at X with $Xa6, Xb7$ collinear in that order (as to block nests $6,7$). This means that $ab76$ is a convex quadrilateral, and thus the other intersection of $a7, b6$ is contained within the quadrilateral.

However this new intersection would be between $a, 7$ and $b, 6$, so the nests are not blocked. Therefore at most 1 such point is possible for each pair a, b. Creating a construction that satisfies this for each pair is easy.

Problem 6

In a deck of 20 cards, 15 are black, 5 are white. This deck is randomly arranged. A *group* is a set of consecutive cards of the same color such that on either end is the opposite color or an end of the deck. What's 4 times the expected number of *groups*?

Proposed by Milo Stammers

Solution

Consider the pairs of consecutive cards, $1-2$, $2-3$,...,$19-20$. The number of groups will equal 1 more than the number of these pairs that have different colors. For any pair, the probability that it has opposite colors is $2 * \frac{5 \cdot 15}{20 \cdot 19} = \frac{15}{38}$. Applying linearity of expectation, the number of groups is expected to be $1 + 19 * \frac{15}{38} = \frac{17}{2}$. The answer is $\boxed{34}$.

Problem 7

The numbers from 1 to $10,000,000$ are written on a whiteboard, each on a separate line. How many times does the digit 0 appear directly before the digit 1? (The numbers are written without leading zeroes).

Proposed by Alex Wang, Solution by Milo Stammers

Solution

Ignore $10,000,000$ because it does not have any 01's. First consider if we did allow leading 0's. Then we are essentially considering all strings of digits length 7. If we put an 01 in some position, there are 10^6 ways to pick the rest of the digits. There are 6 places to put the 01. Note that some numbers have multiple 01 strings, but we are counting it for exactly how many times it occurs, as we want. This gives us $6 * 10^6$ occurrences.

Now removing the leading 0's, we subtract one for each number starting with 1 that has leaving $0's$. If the number begins 01..., there are 10^5 ways to fill in the rest of the number, if it begins 001..., then there are 10^4 ways, and so on. Thus our final answer is $6 * 10^6 - 10^5 - 10^4 - .. - 10^0 = \boxed{5888889}$.

Problem 8

2024 people are standing in a line. The first person is holding a ball. On each turn the person holding the ball passes the ball to someone ahead in the line, each person equally likely. The probability that the 2000'th person holds the ball at some point can be written as m/n where m and n are relatively prime positive integers. What is $m + n$?

Proposed by Milo Stammers

Solution

Consider the last 25 people, numbered $2000-2024$ inclusive. Eventually the ball will reach this range of people, and when it does it will be equally likely to first be passed to any one of them. However, if it is passed to anybody past 2000, then they will never get the ball, so it must first reach person 2020 when it enters the final 25 people. The probability of the happening will thus be $\frac{1}{25}$, and the answer is $\boxed{26}$.

Alternatively, you may have set up a recursion formula for the probability the ball reaches the 25'th last person, and you will get this comes out to the constant $\frac{1}{25}$ once again.

Problem 9

Find the number of bijective functions f from the set $\{1,2,3,4,5,6\}$ to itself such that $f(f(f(f(x))))=x$ for all x.

Proposed by Milo Stammers, Solution by Anna Zhou

Solution

We consider cycles $a_1,a_2,...a_n$ such that $f(a_1)=a_2, f(a_2)=a_3,\dots,f(a_n)=a_1$. Since we apply the function 4 times, the length of each cycle must be a factor of 4. We perform casework on the length of the longest cycle.

Case 1: All cycles have length 1. We get 1 for this case.

Case 2: Longest cycle has length 2. We have 3 subcases.

Subcase 2a: Cycles have length 1, 1, 1, 1, and 2. We have $\binom{6}{2}=15$ for this case.

Subcase 2b: Cycles have length 1, 1, 2, and 2. We have $\binom{6}{2}*\binom{4}{2}/2=45$ for this case.

Subcase 2c: Cycles have length 2, 2, and 2. We have $\binom{6}{2}*\binom{4}{2}/6=15$ for this case.

Case 3: Longest cycle has length 4. We have 2 subcases.

Subcase 3a: Cycles have length 1, 1, and 4. We have $\binom{6}{4}*3!=90$ for this case.

Subcase 3b: Cycles have length 2 and 4. We have $\binom{6}{4}*3!=90$ for this case.

Summing all of our cases, we have $\boxed{256}$ as our answer.

Problem 10

An equilateral triangle of side length 3 is split into 9 unit equilateral triangles. Each of these 9 triangles is given colored red, green, or blue. How many ways are there to color the triangles where rotations and reflections are considered equivalent?

Proposed by Milo Stammers

Solution

To deal with our worry about rotations and reflections, we will apply Burnside's Lemma. The number of symmetries for the triangle is 6: do nothing, rotate left, rotate right, and reflections about each axis of symmetry. All 3^9 triangles are fixed by doing nothing. For a rotation, there are three groups of subtriangles that must all be the same color (the tips, then alternating triangles in the center). For the reflections, there are now 6 groups, the three individual triangles in the middle, then three pairs with a triangle either side. Altogether, with Burnside's Lemma we get the answer to be $\frac{1}{6}[3^9+2*3^3+3*3^6]=\boxed{3654}$.

Problem 11 (Tiebreaker)

Consider a bijection from $1, 2, ..., 2024$ to itself. Let N be the expected value for the minimum n such that the n'th iteration $f^n(x) = x$ for all x. Estimate $\log_{10}(N)$.

Proposed by Milo Stammers

Solution

n is known as the order of the permutation f. See this paper by Goh and Schmutz researching the limiting behavior for the expected value of the order of a permutation in the symmetric groups.

Goh, W. M. Y., Schmutz, E. (n.d.). The expected order of a random permutation. https://www.math.drexel.edu/ eschmutz/PAPERS/musn.pdf

Applying the results, $\log_{10}(N) \approx \boxed{21.177}$.