PVMT 2024: Algebra and Number Theory Round Solutions

Problem 1

A bucket of water starts out at 10 liters. Each day, due to evaporation, the volume is reduced by 20%. The number of liters in the bucket after 3 days can be represented as a/b, where a and b are relatively prime positive integers. What is $a+b$?

Proposed by Alyssa Yu

Solution

Each day, the volume of water is multiplied by $80\% = \frac{4}{5}$. Thus after 3 days, we are left with $10 * \frac{4^3}{5^3} = \frac{640}{125} = \frac{128}{25}$. Thus our answer is $128+25 = \boxed{153}$.

Problem 2

Five positive integers $a \geq b \geq c \geq d \geq e$ have an arithmetic mean of 2.4 and a mode (number that appears most often) of 3. What is $a-b+c-d+e$?

Proposed by Alyssa Yu

Solution

We have $a+b+c+d+e = 12$ so if four of the numbers are equal to 3, the last number must be 0. Hence there can either be two or three 3's.

If there are exactly two 3's, then the other three numbers must all be different as to have a unique mode. The only way we can achieve this is to have them equal 1, 2, and 3 which produces a contradiction.

If there are exactly three 3's, then the other two numbers must be 1 and 2. Thus $(a,b,c,d,e) = (3,3,3,2,1)$. The answer is $3-3+3-2+1 = \boxed{2}$.

Problem 3

In reduced form, $\frac{p}{q}$ is the largest number that cannot be written as a sum of numbers in the range $(\frac{4}{10}, \frac{6}{10})$. What is $p+q$?

Proposed by Milo Stammers

Solution

With two numbers in the range, we can get any value in $(\frac{8}{10}, \frac{12}{10})$, the lower end being picking two values as small as possible and the upper being two values as large as possible. With three numbers, similarly, we can get in the range $(\frac{12}{10}, \frac{18}{10})$, then with four $(\frac{16}{10}, \frac{24}{10})$. See that $\frac{16}{10} < \frac{18}{10}$ and the intervals begin overlapping and thus removing any gaps in possible numbers. Therefore, the answer is $\frac{12}{10} = \frac{6}{5}$, which is just impossible for 2 or for 3 numbers. This gives us $\boxed{11}$.

Problem 4

$2m$ students from Euclid HS and $3m$ students from Euler HS participated in PVMT, for a certain positive integer m. Unfortunately, $14-m$ students from Euclid HS and $5m-11$ from Euler HS flunked the Algebra and Number Theory round. How many students came from Euclid HS?

Proposed by Anna Zhou

Solution

Because the number of students who flunked must be less than or equal to the total students from the school, we have $0 \leq 14-m \leq 2m$ and $0 \leq 5m-11 \leq 3m$. The first inequality gives us $m \geq \frac{14}{3}$ and the second $m \leq \frac{11}{2}$. For integers, the first inequality tells us that m is at least 5 and the second that it must be less than 6, so $m=5$ is the only possibility. This gives the answer $2(5) = \boxed{10}$.

Problem 5

Spunk the spider and Flaia the fly are racing down a 560-inch track. Each creature's crawling speed is directly proportional to the number of legs they have. If Spunk beats Flaia to the finish line, he will eat her up. Fortunately, Flaia can fly and her flying speed is 12 times her crawling speed. But if Flaia beats Spunk, the spiteful spider will eat her anyway. The fraction of the track that Flaia must fly to ensure that she and Spunk cross the finish line at the same time can be represented as m/n, where m and n are relatively prime positive integers. What is $m \cdot n$?

Proposed by Anna Zhou

Solution

Since the problem is asking for a fraction, the length of the track doesn't matter, so we can just make it 1. Similarly, the creatures' exact speeds don't matter either; we can make them 8, 6, and 72 for Spunk's crawling, Flaia's crawling, and Flaia's flying speeds, respectively. Let x be the fraction of the race that Flaia flies. Now we just need to find the amount of time each creature takes to get to the finish line and set them equal to each other. Equating times as distance over speed, we get $\frac{1}{8} = \frac{1-x}{6} + \frac{x}{72}$. Solving, we get $x = \frac{3}{11}$, so the answer is $3 \cdot 11 = \boxed{33}$.

Problem 6

Let n be smallest integer greater than $30!+1$ that does not have any factors less than or equal to 30 besides 1. Find $n - 30!$.

Proposed by Logan Van Pelt

Solution

For $k = 2,3,4,\ldots,30$, we can factor k out of $30!+k = k(1 \cdot 2 \cdot \ldots \cdot (k-1) \cdot (k+1) \cdot \ldots \cdot 30 + 1)$, thus the number has a factor $k \leq 30$ not equal to 1. However, notice that 31 is prime, so we take $30!+31 \equiv 310 \mod k$ because k cannot divide 31. Thus there is no factor less than 30 besides 1 that divides $30!+31$, and our answer is $\boxed{31}$.

Problem 7

Find the largest $n \leq 200$ that is a multiple of 10 such that $n^2 - n \equiv 3 \pmod 9$.

Proposed by Ricky Sun

Solution

Testing values of n from 0 to 8, we find the only solutions are $n = 4,6 \mod 9$. Then notice that $10n \equiv n \equiv 4,6 \mod 9$, so just solve for such $n \leq 20$. The maximum value is 15 so multiply by 10 to get $\boxed{150}$.

Problem 8

Define $f(z) = z^2 - 2$ for complex z. Let S be the set of complex numbers z such that $f^n(z) = f(f(f(...f(z)...)))$ is bounded for arbitrarily large n. Find the area of the smallest square that contains S.

Proposed by Milo Stammers

Solution

For any z in the complex plane, we may write it as some $u + \frac{1}{u}$. Substituting this, $f(z) = (u+\frac{1}{u})^2 - 2 = u^2 + \frac{1}{u^2}$. Iterating again, $f(f(z)) = (u^2 + \frac{1}{u^2}) - 2 = u^4 + \frac{1}{u^4}$. We can continue this to see $f^n(z) = u^{2^n} + \frac{1}{u^{2^n}}$. If $|u| > 1$, then the first term will go to infinity and the second to 0, thus would not work. If $|u| < 1$, then the first term would go to 0 and the second to infinity. But when $|u| = 1$, both terms maintain unit magnitude, so are bounded. Therefore, the possible values of z can be written as $(\cos(\theta) + i\sin(\theta)) + \frac{1}{(\cos(\theta)+i\sin(\theta))} = \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta) = 2\cos(\theta)$, where $u = \cos(\theta) + i\sin(\theta)$. Thus $S = [-2,2]$ on the real line. The longest line segment in a square is its diagonal, so the smallest square containing S has vertices at $2, -2, 2i, -2i$ with area $\boxed{8}$.

Problem 9

What is the smallest positive integer k for which there exists an integer $n \geq 1$ such that the sum of the first n natural numbers is equal to $2024k$?

Proposed by Anna Zhou

Solution

The sum of the first natural numbers is $\frac{n(n+1)}{2} = 2024k$. We want to find the smallest k so that there is an integer solution to $n(n+1) = 2^4 * 11 * 23 * k$. Clearly the two factors are of opposite parity. Accordingly, we divide the problem into 4 cases.

Case 1: The two factors are $16m$ and $11*23*n$, $mn = k$ and n is odd. We need $11*23*n \equiv (-3)*n \equiv \pm 1 \pmod{16}$. Checking gives the smallest value of n as 5. The corresponding

value of m, which is also smallest, is 79. This makes 395 the smallest k for this case.
Case 2: The two factors are $16*11*m$ and $23*n$, $mn = k$ and n is odd. We need $16*11*n \equiv (-8)*m \equiv \pm 1 \pmod{23}$. Checking gives the smallest value of m as 3. The corresponding value of n (we check to make sure it's odd) is 23. This makes 69 the smallest k for this case.
Case 3: The two factors are $16*23*m$ and $11*n$, $mn = k$ and n is odd. We need $16*23*n \equiv 5*m \equiv \pm 1 \pmod{11}$. Checking gives the smallest value of m as 2. The corresponding value of n (we check to make sure it's odd) is 67. This makes 134 the smallest k for this case.
Case 4: The two factors are $16*11*23*m$ and n, $mn = k$ and n is odd. We need $n \equiv \pm 1 \pmod{4048}$. The smallest n is 4047; this is already much bigger than all of our previous answers, so we can stop here.

Looking at all of our cases, we see that the smallest value of k is $\boxed{69}$.

Problem 10

For $x \in [0,1]$, find the expected value of:

$$2^{\sqrt{x}} + [\log_2(x+1)]^2$$

Proposed by Milo Stammers

Solution

Let $f(x) = y = [\log_2(x+1)]^2$, then $2^{\sqrt{y}} - 1 = x$. Thus we wish to find $E(1 + f(x) + f^{-1}(x))$. The expected value of each of these terms is the average height of their graphs over the region, but because the interval is length 1, we just need to find the area beneath them and add the extra 1. Computing each term separately is difficult, but consider the path traced by the function $f(x)$ from $(0,0)$ to $(1,1)$. It is clearly always increasing, and the inverse function is a reflection of $f(x)$ over the line $y = x$. Thus putting the graphs together would make a complete unit square. Thus the sum of the areas is 1, so the total expected value is $1 + 1 = \boxed{2}$.

Problem 11 (Tiebreaker)

As a decimal, estimate the value of

$$\prod_{\text{prime p}} \frac{p^2}{p^2+1}$$

Proposed by Milo Stammers

Solution

Dividing the numerator and denominator of each fraction by p^2 them multiplying the top and bottom by $1 - \frac{1}{p^2}$ we get

$$\frac{\prod_{\text{prime} p} 1 - \frac{1}{p^2}}{\prod_{\text{prime} p} 1 - \frac{1}{p^4}} = \frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{15} = .65797...$$

where $\zeta(z)$ is the Riemann Zeta Function. This problem utilizes the Euler Product form of the Zeta Function, which allows for some very strange and beautiful results such as this one shown.