PVMT 2024: Geometry Round

Problem 1

Square A has vertices $(0,0)$, $(6,0)$, $(0,6)$, and $(6,6)$. Square B has half the area of A and each vertex of B has integer coordinates. If B is fully contained in A, what is the product of the sums of the x and y coordinates of each vertex of B?

Problem 2

Right triangle ABC with right angle at B has perimeter $4+2\sqrt{6}$. Median BD has length 2. What is the area of ABC?

Problem 3

In rectangle $ABCD$, let $AB=3$, $BC=4$. Let M be the midpoint of BC and N the midpoint of CD. The area of the triangle enclosed by AM, BN, and AB can be written as m/n where m and n are relatively prime positive integers. Find m^2+n^2.

Problem 4

Define points B, A, C at $(0,0),(1,0),(2,0)$. Let circle O be a circle with radius 1 centered at A. Let P be on O and let the tangent at P to circle O be line L. Let the closest point on line L to point C be point X. Define point Y such that Y,X,C are collinear in that order and $XC\cdot YC=2$. Suppose Y has a x-coordinate of 1.5. What is $10\cdot YC$?

Problem 5

Triangle ABC has integer side lengths that form an arithmetic progression. The largest angle is twice the smallest angle. Given that one of the sides has length 2024, what is the perimeter of ABC?

Problem 6

In triangle ABC, $AB=2$, and BD is the altitude from B to AC. Suppose A,B,C are chosen such that $BD=\frac{AC}{2}$. The minimum value of BC can be written as $a\sqrt{b}-c$. What is $100a\cdot b\cdot c$?

Problem 7

Let A and B be $(1,5)$ and $(3,1)$, respectively. Let P be a point that is distance 4 away from $(37,15)$. The sum of the coordinates of P when PA^2+PB^2 is maximized can be represented as m/n where m and n are relatively prime positive integers, find $m+n \mod 1000$.

Problem 8

Square $ABCD$ has side length 7. Draw a quarter-circle centered at B going through A and C. Let E be a moving point on this quarter circle. Let F be the point on segment BE such that $BF=3$. The minimum value of $CF+DE$ can be written as $\sqrt{x}$. What is x?

Problem 9

Let $\triangle ABC$ have internal angle bisector AD and external angle bisector AE, where B, D, C, E are collinear in that order. Given that $CE = 8$ and $DE = 10$, the length BE can be written as m/n where m and n are relatively prime positive integers. What is $m+n$?

Problem 10

Let ABC be a triangle where $AB = 5$, $BC = 8$, and $AC = 7$. Let I_B and I_C be the B-excenter and C-excenter of ABC (the excenter is the center of the circle tangent to one side and the extensions of the other two sides. For example the A-excenter is the center of the circle tangent to the extensions of AB and AC and segment BC; the B and C-excenters are defined similarly). The value $I_BI_C \cdot BC + BI_C \cdot CI_B$ can be written as $\frac{20\sqrt{x}}{3}$ where x is a positive integer. Find x.

Problem 11 (Tiebreaker)

A cyclic pentagon $ABCDE$ inscribed in a circle with center O can be divided into 3 kites $OABC$, $OCDM$, and $OMEA$ where M is the midpoint of DE. Estimate the ratio between the area of pentagon $ABCDE$ and the area of circle O. Write your answer in the form $0.abcdef$.