PVMT 2024: Combinatorics Round

Problem 1

Anna picks an edge and a face of a cube at random. What is the probability that they share at least one vertex, rounded to the nearest percent?

Problem 2

Soham starts at $(0,0)$ and randomly chooses with equal probability of $\frac{1}{5}$ between moving 1 unit left, 2 units down, 3 units right, 4 units up, and stopping. Soham keeps doing this until he stops. What is the expected value of the sum of the x and y coordinates that Soham stops at?

Problem 3

Take a set of positive integers $S = \{a_1, a_2, ..., a_7\}$. Let A, B be disjoint subsets of S. Find the largest n such that all positive integers from 1 to n can be expressed as the sum of the elements in A minus the sum of the elements in B with some fixed choice of S.

Problem 4

Kabosu farms 1 dogecoin per day, Monday through Thursday. On Friday Kabosu either triples her dogecoin with probability $\frac{1}{4}$ or sells all of her dogecoin with probability $\frac{3}{4}$. She does nothing on the weekend. Kabosu repeats this until the end of time. After a long, long time, what is the expected number of dogecoin she has?

Problem 5

The PVMT team visits a park with seven falcon nests, numbered 1-7. After a while, everyone says they can see nests 1-5 but not nests 6 or 7. The only way you can't see a nest is if another nest is directly blocking it. No three nests are collinear and everything is on the same plane. No human is at a nest and no two humans occupy the same spot. At most how many members are on the PVMT team?

Problem 6

In a deck of 20 cards, 15 are black, 5 are white. This deck is randomly arranged. A *group* is a set of consecutive cards of the same color such that on either end is the opposite color or an end of the deck. What's 4 times the expected number of *groups*?

Problem 7

The numbers from 1 to $10,000,000$ are written on a whiteboard, each on a separate line. How many times does the digit 0 appear directly before the digit 1? (The numbers are written without leading zeroes).

Problem 8

2024 people are standing in a line. The first person is holding a ball. On each turn the person holding the ball passes the ball to someone ahead in the line, each person equally likely. The probability that the 2000'th person holds the ball at some point can be written as m/n where m and n are relatively prime positive integers. What is $m+n$?

Problem 9

Find the number of bijective functions f from the set $1,2,3,4,5,6$ to itself such that $f(f(f(f(x)))) = x$ for all x.

Problem 10

An equilateral triangle of side length 3 is split into 9 unit equilateral triangles. Each of these 9 triangles is given colored red, green, or blue. How many ways are there to color the triangles where rotations and reflections are considered equivalent?

Problem 11 (Tiebreaker)

Consider a bijection from $1,2,\ldots,2024$ to itself. Let N be the expected value for the minimum n such that the n'th iteration $f^n(x) = x$ for all x. Estimate $\log_{10}(N)$.