PVMT 2024: Algebra and Number Theory Round

Problem 1

A bucket of water starts out at 10 liters. Each day, due to evaporation, the volume is reduced by 20%. The number of liters in the bucket after 3 days can be represented as a/b, where a and b are relatively prime positive integers. What is $a+b$?

Problem 2

Five positive integers $a \geq b \geq c \geq d \geq e$ have an arithmetic mean of 2.4 and a mode (number that appears most often) of 3. What is $a-b+c-d+e$?

Problem 3

In reduced form, $\frac{p}{q}$ the largest number that cannot be written as a sum of numbers in the range $(\frac{4}{10}, \frac{6}{10})$. What is $p+q$?

Problem 4

$2m$ students from Euclid HS and $3m$ students from Euler HS participated in PVMT, for a certain positive integer m. Unfortunately, $14-m$ students from Euclid HS and $5m-11$ from Euler HS flunked the Algebra and Number Theory round. How many students came from Euclid HS?

Problem 5

Spunk the spider and Flaia the fly are racing down a 560-inch track. Each creature's crawling speed is directly proportional to the number of legs they have. If Spunk beats Flaia to the finish line, he will eat her up. Fortunately, Flaia can fly and her flying speed is 12 times her crawling speed. But if Flaia beats Spunk, the spiteful spider will eat her anyway. The fraction of the track that Flaia must fly to ensure that she and Spunk cross the finish line at the same time can be represented as m/n, where m and n are relatively prime positive integers. What is $m \cdot n$?

Problem 6

Let n be smallest integer greater than $30!+1$ that does not have any factors less than or equal to 30 besides 1. Find $n-30!$.

Problem 7

Find the largest $n \leq 200$ that is a multiple of 10 such that $n^2 - n \equiv 3 \pmod 9$.

Problem 8

Define $f(z) = z^2 - 2$ for complex z. Let S be the set of complex numbers z such that $f^n(z) = f(f(f(...f(z)...)))$ is bounded for arbitrarily large n. Find the area of the smallest square that contains S.

Problem 9

What is the smallest positive integer k for which there exists an integer $n \geq 1$ such that the sum of the first n natural numbers is equal to $2024k$?

Problem 10

For $x \in [0,1]$, find the expected value of:

$$2^{\sqrt{x}} + [\log_2(x+1)]^2$$

Problem 11 (Tiebreaker)

As a decimal, estimate the value of

$$\prod_{\text{prime p}} \frac{p^2}{p^2+1}$$