## Problem 1

For real numbers $x > y$, we are given $(x-y)^2 = 18$ and $(x+y)^2 = 8$. Evaluate $x^2 - y^2$.

## Solution

Multiply the equations to get

$$(x-y)^2(x+y)^2 = 144$$
$$(x^2-y^2)^2 = 144$$
$$x^2 - y^2 = \boxed{12}$$

*Proposed by Milo Stammers*

## Problem 2

Woody the woodchuck is chucking wood. He chucks 1 piece of wood on the first day, 2 pieces on the second day, 4 pieces on the third day, and so on, doubling each day. If the forest has 2023 pieces of wood, how many days does it take for Woody to chuck the entire forest? If there are not enough pieces of wood, then Woody will simply chuck the rest.

## Solution

Counting how much wood has been chucked after each day, 1, 3, 7, 15, 31, 63, 127... The sum after $n$ days is $2^n - 1$ so we have to find the least power of 2 greater than 2023. This can be easily proven with induction because $(2^n - 1) + 2^n = 2^{n+1} - 1$. $1024 < 2023 < 2048$, 2048 is $2^{11}$ so it will take Woody $\boxed{11}$ days to chuck the forest.

*Proposed by Daisy Dastrup*

## Problem 3

What is the remainder when $2^{3^{4^5}}$ is divided by 7?

## Solution

Note: $a^{b^c}$ means $a^{(b^c)}$ Looking at the remainders of powers of 2 modulo 7:

$$2, 4, 1, 2, 4, 1...$$

A cycle of $2-4-1$ repeating every 3. Since the power $3^{4^5}$ is a multiple of 3, we would gte our answer to be equal to $2^3$ mod 7, which is equal to $\boxed{1}$.

*Proposed by Alyssa Yu*

## Problem 4

We are given the two sums of infinite geometric series:

$$a + ar + ar^2 + ar^3 ... = 23$$

$$a - ar + ar^2 - ar^3 ... = 20$$

Compute the value of

$$a^2 + a^2r^2 + a^2r^4 + ...$$

## Solution

Use the infinite geometric series formula to rewrite the equations

$$\frac{a}{1-r} = 23$$

$$\frac{a}{1+r} = 20$$

and we are looking for

$$\frac{a^2}{1-r^2}$$

If we multiply the two given equations and apply different of squares we get

$$\frac{a}{1-r}\frac{a}{1+r} = \frac{a^2}{1-r^2} = 20 * 23 = \boxed{460}$$

*Proposed by Milo Stammers*

## Problem 5

There is a glass that is 2023 meters tall. I fill the bottom 1 meter with $\frac{1}{3}$ concentration lemonade (1 part lemon for 2 parts water). I fill the next $\frac{1}{2}$ meter with $\frac{1}{4}$ concentration lemonade. I continue this, on the $n$'th step, I fill $\frac{1}{n}$ meters with $\frac{1}{n+2}$ concentration lemonade. I do this until the glass would overflow (which will happen). Let $C$ be the final volume of lemonade in the glass (assuming bottom area to be 1 square meter), rounded to the nearest hundredth. Find $100C$.

## Solution

On the first step, we are adding $\frac{1}{1}\frac{1}{3}$ meters of lemonade, and in general on the $k$'th step we are adding $\frac{1}{k}\frac{1}{k+2}$ meters of lemonade. Let $m$ be the step we stop, we know that $m$ is the largest integer such that

$$\frac{1}{1} + \frac{1}{2} + ...\frac{1}{m} < 2023$$

which is very large, and our answer does not need high precision.

The total lemonade will be

$$\Sigma_{i=1}^{m}\frac{1}{n(n+2)} = \frac{1}{2}\Sigma_{i=1}^{m}(\frac{1}{n} - \frac{1}{n+2})$$

where we broke apart the fraction using partial fractions decomposition. Writing this out

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}...+\frac{1}{m-1}-\frac{1}{m+1}+\frac{1}{m}-\frac{1}{m+2}=$$

$$\frac{1}{1}+\frac{1}{2}-\frac{1}{m+1}-\frac{1}{m+2}$$

Because $m$ is so large we can approximate the last terms to be zero as we only need to the nearest hundredth

$$\frac{1}{2}[\frac{1}{1}+\frac{1}{2}]=.75$$

finally, we have that $100C=\boxed{75}$.

*Proposed by Milo Stammers*

## Problem 6

Consider three reals $a,b,c>0$ such that $ab^2c^3=2$. The cube of the minimum possible value of

$$\frac{a}{b^2c^3}+\frac{b^2}{ac^3}+\frac{c^3}{ab^2}$$

is $\frac{p}{q}$ for relatively prime positive integers $p,q$. What is $p+q$?

## Solution

First, rewrite the fractions to all have the same denominator and combine,

$$\frac{a^2+b^4+c^6}{ab^2c^3}=\frac{a^2+b^4+c^6}{2}$$

By the AM-GM inequality, we know that $\frac{a^2+b^4+c^6}{3}\geq\sqrt[3]{a^2b^4c^6}=\sqrt[3]{4}$ With equality when $a^2=b^4=c^6$. Therefore the minimum value of $a^2+b^4+c^6=3\sqrt[3]{4}$, and the minimum value of the original expression is $\frac{3}{\sqrt[3]{2}}$, cubed $\frac{27}{2}$, so the answer is $27+2=\boxed{29}$.

*Proposed by Milo Stammers*

## Problem 7

Let

$$f(x)=\sqrt{x^2+1\sqrt{x^4+2\sqrt{x^8+3\sqrt{...\sqrt{x^{2^n}+n\sqrt{...}}}}}}$$

## Solution

Find $f^{-1}(f^{-1}(2024))$ given that $\sqrt{1+2\sqrt{1+3\sqrt{...}}}=3$. Consider the first few radicals in $f(x)$

$$\sqrt{x^2+1\sqrt{x^4+2\sqrt{x^8+...}}}=$$

$$\sqrt{x^2+1\sqrt{x^4+2x^4\sqrt{1+...}}} =$$

$$\sqrt{x^2+1x^2\sqrt{1+2\sqrt{1+...}}} =$$

$$x\sqrt{1+1\sqrt{1+2\sqrt{1+...}}}$$

We can factor an $x$ out of the radical to reduce the radical to pure numeric form. You can see that this works by multiplying $x$ into the radical above.

Now we can evaluate this

$$\sqrt{1+1\sqrt{1+2\sqrt{1+3\sqrt{...}}}} =$$

$$\sqrt{1+1(3)} = \sqrt{4} = 2$$

Thus $f(x) = 2x$, $f^{-1}(x) = \frac{x}{2}$ so

$$f^{-1}(f^{-1}(2024)) = f^{-1}(\frac{2024}{2}) = \frac{2024}{4} = \boxed{506}$$

*Proposed by Milo Stammers*

## Problem 8

Find the sum of all positive integers $n \leq 100$ for which $n^2+n+31$ is divisible by 43.

## Solution

Apply quadratic formula, so $n \equiv \frac{-1\pm\sqrt{-123}}{2} \pmod{43}$. $\sqrt{-123}$ is not the regular square root over the reals, it is the number such that when squared is congruemnt to $-123$ modulo 43. Substitute $-123$ with $-123+5*43 = 49$. So $n \equiv \frac{-1\pm 7}{2} \pmod{43}$. Then $n \equiv 3, -4 \mod 43$. So $n$ can equal $3, 39, 46, 82, 89$. Sum: $\boxed{259}$

## Problem 9

Let function $f(x)$ satisfy for $x \neq 0, -1$:

$$\frac{f(x)+1}{x+1} = \frac{f(x+1)}{x}$$

And let $\frac{1}{x} \leq f(x) \leq \frac{2}{x}$ for $0 < x < 1$. Suppose that the difference between the maximum and minimum possible values for $f(2023.6)$ can be written as $\frac{p}{q}$, for relatively prime positive integers $p, q$. Compute $p+q$.

## Solution

Cross multiply the fractions to get that

$$xf(x)+x=(x+1)f(x+1)$$

$$x=(x+1)f(x+1)-xf(x)$$

Now substitute $xf(x)=g(x)$ to get

$$g(x+1)-g(x)=x$$

Now if we take $x=n+r$ where $n$ is an integer and $0\leq r<1$, then by repeated application of the above identity

$$g(x)=g(n+r)=g(r)+r+(1+r)+...(n-1+r)=g(r)+nr+\frac{n(n-1)}{2}$$

Also, for $0<x<1$, we know that $\frac{1}{x}\leq f(x)\leq\frac{2}{x}$ so $1\leq g(x)\leq 2$. Thus converting back to $f(x)$,

$$f(x)=\frac{g(r)+nr+\frac{n(n-1)}{2}}{n+r}$$

Letting $x=2023.6$,

$$f(2023.6)=\frac{g(.6)+(.6)2023+\frac{2023(2022)}{2}}{2023.6}$$

Because we only need the difference between minimum and maximum values, we can avoid most of the computation. The maximum is achieved when $g(.6)=2$, and the minimum is when $g(.6)=1$, taking the difference we find our answer to be

$$\frac{2}{2023.6}-\frac{1}{2023.6}=\frac{1}{2023.6}=\frac{5}{10118}$$

So the answer is $5+10118=\boxed{10123}$.

*Proposed by Milo Stammers*

## Problem 10

Consider the sequence of positive reals starting with $a_2=\frac{1}{4}$ and

$$a_n{}^{\frac{1}{n}}((n+1)!-1)=((n+1)!-n-1)a_{n+1}{}^{\frac{1}{n+1}}$$

## Solution

Find the smallest $n$ such that $a_n>1-\frac{1}{2023}$. Consider the sequence $b_n=a_n{}^{\frac{1}{n}}$, then we have that $b_2=\frac{1}{2}$ and our recurrence becomes

$$\frac{b_n}{(n+1)!-n-1}=\frac{b_{n+1}}{(n+1)!-1}$$

$$\frac{b_n}{1-\frac{1}{n!}}=\frac{b_{n+1}}{1-\frac{1}{(n+1)!}}$$

Thus $\frac{b_n}{1-\frac{1}{n!}}$ is constant.

$$\frac{b_n}{1-\frac{1}{n!}} = \frac{b_2}{1-\frac{1}{2!}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$b_n = 1 - \frac{1}{n!}$$

$$a_n = {b_n}^n = (1 - \frac{1}{n!})^n$$

Now we can use the binomial theorem to evaluate $a_n$

$$a_n = (1 - \frac{1}{n!})^n = 1 - \frac{n}{n!} + \frac{n(n-1)}{2}\frac{1}{n!^2} + ...$$

The terms after the first two are incredibly small, because you are raising $n!$ to increasingly large powers. Because we only need to know when $a_n$ becomes greater than $1 - \frac{1}{2023}$, we can use the first two terms

$$1 - \frac{n}{n!} = 1 - \frac{1}{(n-1)!}$$

Thus we want $2023 < (n-1)!$. The smallest factorial greater than 2023 is $5040 = 7!$. Thus $n-1 = 7$, $n = \boxed{8}$.

*Proposed by Milo Stammers*