PVMT 2023: Team Round

Problem 1

Poolesville is sending 10 students to a math competition, two teams of 5. How many ways are there to create the two teams (The order of the teams do not matter, only who is on the teams)?

Problem 2

Let there be a set with positive integers, $\{1,2,3,20,22,23\}$ and let a,b,c,d,e,f be distinct elements from the set.

Find the minimum value of the expression.

$\left\lceil \frac{ab}{cd} \right\rceil^{\left\lceil \frac{a+b}{e+f} \right\rceil} + \left\lfloor \frac{e}{f} \right\rfloor^{\left\lceil \frac{e+f}{a+b} \right\rceil}$

Note: $\lceil x \rceil$ is the least integer greater than x, and $\lfloor x \rfloor$ is the greatest integer less than x

Problem 3

Consider the points $A=(5,0)$, $B=(1,1)$, $C=(0,6)$, $D=(-1,1)$, $E=(-7,0)$, $F=(-1,-1)$, $G=(0,-8)$, $H=(1,-1)$. What is the area of polygon $ABCDEFGH$?

Problem 4

Evaluate

$$(\log_3 94 \log_{94} 7 \log_7 3)(144^{\log_{12} 21})$$

Problem 5

How many ways to arrange the string "122345" such that no number goes into its original position (either of the 2's cannot go into positions 2 or 3.)

Problem 6

The surface area of a non-degenerate rectangular prism is 12. You are now given the sum of the lengths of the edges. With this information, the volume is uniquely determined. What is the square of the numerical sum of all the lengths of the edges of the prism and the volume of the prism?

Problem 7

How many sequences of length 8 formed by the letters A,B, and C are there, where A appears an even (possibly 0) number of times, B appears an odd number of times, and C appears a multiple of 3 times?

Problem 8

Line segment AB forms a 60° angle with line segment AC. Given that $AB = 2$ and $AC = 2$, the area of the region, R, where R contains all such points P on the plane of the figure where there exists some point X on line segments AB or AC such that $AX = 2 * PX$, can be expressed as $\frac{a}{b}\pi + c\sqrt{d}$ for relatively prime integers a and b and simplified radical with integers $c\sqrt{d}$. Find $a + b * c + d$

Problem 9

In an equilateral triangle ABC let the mid point of BC be denoted as M. A point P is placed in triangle AMC and altitudes are dropped from P to sides BC and AC at point A' and B' respectively. Given that $PA = \sqrt{13}$, $PB' = 1$, and $PA' = 7$, the area of triangle AMP can be represented as $\frac{a\sqrt{c}}{b}$ for relatively prime integers a and b and simplified radical $a\sqrt{c}$. What is $a + b + c$?

Problem 10

Let $ABCDEF$ be a regular hexagon with side length 2, and M the midpoint of AB. The angle bisector of $\angle MDE$ intersects AF at Q. Find $FQ = a\sqrt{b} - c$ for squarefree b, what is $ab + c$?

Problem 11

$ABCD$ is a rectangle with $AB = 8$ and $BC = 5$. Extend AD to P and CB to Q such that P and Q are on opposite sides of AB, $PD = 3$, and $BQ = 6$. The segment PQ intersects AB at E and CD at F. The length of EF can be written as $\frac{m}{n}\sqrt{k}$, where m and n are coprime, and k is not divisible by the square of any integer. What is $m + n + k$?

Problem 12

Find the maximum of $3x + 4y$ if $x^2 - 4x + y^2 - 6y = 12$.

Problem 13

Three points A, B, C are chosen in circle O with radius 10 such that ABC is acute. A circle X in O is mutually tangent to Line BC and the minor arc BC. Another circle Y that is not X in O is drawn that fulfills the same conditions, and is drawn to intersect X in two points, P and Q. If $AB = 15, BC = 18$, and ray PQ goes through B, find the length CA.

Problem 14

The minimum area of a circumscribed ellipse around a 5-12-13 right triangle is $\frac{a\pi}{\sqrt{b}}$ for squarefree b, what is ab?

Problem 15

The positive real number a produces the minimum value of

$$\sqrt{a^8 + a^4 - 10a^2 + 25} + \sqrt{a^8 - 18a^4 + a^2 + 81}.$$

a can be written as $\frac{w + x\sqrt{y}}{z}$ for squarefree y and $\gcd(w, x, z) = 1$. What is $w + x + y + z$?

Problem 16

Let ABC be a triangle where $AB = 5$, $BC = 7$, and $AC = 8$. Also, let P be a point on AC such that $AP = 5$, and let D be the reflection of A over line BC. The length PD can be expressed as $\frac{m}{n}$ where m,n are integers and $\gcd(m,n) = 1$. Find $m+n$.

Problem 17

Let f be a bijective function with domain and range $\{1,2,3,4,5,6\}$. How many such f are there such that $f(x)+f(f(7-x)) = 7$ for every x in the domain?

Problem 18

Evaluate

$$\Sigma_{i=2}^{\infty} \log \frac{p_i p_{i+1}}{(p_i+1)(p_{i-1}-1)}$$

where p_i denotes the i'th prime number, so $p_1 = 2, p_2 = 3, \ldots$. Your answer will be of the form $\log \frac{a\pi^b}{c}$ where a,c are relatively prime. What is $100a+10b+c$?

(Hint: $\Sigma_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

Problem 19

Let σ_n be a permutation of the set $\{1,2,3,...n\}$, an ordering of the elements of the set in some way. A cycle in a permutation is a tuple of integers $(x_1,x_2,...,x_k)$ such that x_1 maps to x_2, x_2 to x_3, ..., and x_k to x_1. The positive difference in the expected number of cycles in σ_n for $n = 99$ and $n = 101$ is equal to $\frac{p}{q}$ for some relatively prime p,q . What is $p+q$?

Note: the elements can be rotated $x_1 \to x_2 \to ... \to x_k \to x_1$ to create a new tuples, but we will consider this the same cycle.

Problem 20

Let O be the circumcenter of triangle ABC. Let M be the intersection of the tangents at B and C, D be the intersection of AM and BC, and E be the intersection of (ABC) and AM. Also, let X be the foot of O onto AM, and N be the foot of O onto BC.

Given that $BC = 10, BE = 4, MX = 9, MO = \sqrt{107}, OD = 5$ and $CE = AB$, find CE.

(ABC) denotes the circumcircle of ΔABC