PVMT 2023: Geometry Round

Note: For the questions that ask you to express your answers in terms of fractions/radicals, it is possible for some of the terms (e.g. the denominator or the term inside the radical) to be 1.

Problem 1

The diagonal length of the face of a cube is $\sqrt{2023}$ meters long. What is the surface area of the cube?

Problem 2

An octahedron has side length of 1. What is the square of the distance between two opposite vertices?

Problem 3

A circle of radius 1 and center $(0,0)$ is drawn. It is then stretched by a factor of 3 along the line $x=y$, then stretched by a factor of $\frac{1}{2}$ along the line $y=-x$. The points that on on the original and transformed circle form a convex polygon with area $\frac{a\sqrt{b}}{c}$ where a,c are relatively prime and b is squarefree. What is $a+b+c$?

Problem 4

Let $\mathbf{R}$ be the set of points on the plane that satisfy

$$(x^2+y^2-1)(x^2-3y^2)=0$$

$\mathbf{R}$ splits the plane into some of bounded regions and some of unbounded regions (go off to infinity). The difference between the largest and smallest bounded area is equal to $\frac{a\pi}{b}$, what is $a+b$?

Problem 5

Let ABC be a triangle where $AB=10$ and $BC=15$. Let circle S have diameter BC, and X and Y be the intersections of S and AB and AC, respectively. Given that $XB=6$, the area of ΔAXY can be written as $\frac{a\sqrt{b}}{c}$ for squarefree b, what is $a+b+c$?

Problem 6

Let ΔABC have $AB=5$, $AC=8$. Also let the angle bisector from A and the altitude from C intersect at X, and the foot of the altitude from C hit AB at D. The altitude from X to AC intersects AB at E. Given that ΔAXD and ΔEXD are congruent, find BC.

Problem 7

In cyclic quadrilateral $ABCD$, let AC and BD intersect at point E and the diagonals AC and BD are perpendicular. Point F is the midpoint of side AB, G is the midpoint of side CD, and N is the midpoint of side AD while M is the midpoint of side BC. Given that $\angle CEF = 120^\circ$ and $EF=6$, $DG=4$, the area of the quadrilateral is $a+b\sqrt{c}$, with c being squarefree. What is $a+b+c$?

Problem 8

Five regular tetrahedra, all of edge length 1, are arranged so that there is one central tetrahedron with one outer tetrahedron attached to each of its faces (so that their faces coincide). Each outer tetrahedron has an outer vertex. Lines are drawn between these vertices, forming a larger regular tetrahedron. The edge length of the larger tetrahedron is $\frac{p}{q}$, what is pq?

Problem 9

Let $ABCDEFG$ be a heptagon such that the heptagon made by connecting the midpoints of AB, BC, CD, etc has area 270. Let P be an arbitrary point such that areas $PAB + PBC + PCD + PDE + PEF + PFG + PGA$ add up to 100. Denote the centroids of PAB, PBC, PCD, PDE, etc as X_1, X_2, X_3, X_4....

What is the area of $X_1X_2X_3X_4X_5X_6X_7$?

Problem 10

Let ABC be a triangle inscribed in a circumcircle with radius 4 and with center O, and have incenter I. Let the tangents to the circumcircle at B and C intersect at K. If $\angle BAC = 60$, the perimeter of ABC is 16, and the product of the side lengths of ABC is 128, find KI^2.