PVMT 2023: Combinatorics Round

Problem 1

A fair coin is tossed 5 times. The probability that there are exactly 3 heads and 2 tails can be represented as a fraction $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Problem 2

In a basketball game with players A, B, and C, player A scored an even number of points, and player B scored a perfect square number of points. The total number of points scored in the game was 20. Let a be the number of points player A scored, b be the number of points player b scored, and c be the number of points player c scored. How many possible ordered triples (a,b,c) are there?

Problem 3

Jimmy picks two numbers at random from the first 70 positive integers. The probability that the sum of his numbers is a perfect cube can be represented as a fraction $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Problem 4

There are 15 students in a class, 8 girls and 7 boys. A president and 2 vice-presidents must be chosen, such that at least 1 boy and at least 1 girl are chosen. Assuming the two vice-president positions are indistinguishable, how many ways can this be done?

Problem 5

Caleb the coin flipper continuously flips a fair standard coin until he gets two heads in a row. At this point, he flips the coin again to make a decision. If he lands on tails he will continue his game until he reaches two heads again, upon which he will stop. If he lands on heads he will stop his game there. What is the expected number of flips Caleb will make?

Problem 6

Milo the Magician knows 4 goblins and has 5 pieces of candy. Once Milo has distributed all of his candy amongst his goblin friends, each goblin has a $\frac{1}{2}$ chance that they eat eat a candy immediately, unless they have no candy. The expected total number of uneaten candies can be represented as a simplified fraction $\frac{a}{b}$ where a and b are positive integers. What is $a+b$?

Problem 7

How many total triangles are formed when all sides and diagonals are drawn in a regular hexagon? (These triangles are allowed to have diagonal intersections as vertices.)

Problem 8

An ant starting at the bottom left corner of a table modeled by a 2 by 2 grid wishes to meet its colony, which lives on the top edge and right edge. Every minute the ant randomly moves either left, right, up, or down along an adjacent edge to its next point. Due to its sticky feet the ant cannot fall off the table but can wrap around the grid and follow the gridlines from underneath the table(wrapping around does not count as a move). For example, if the ant is in the center of the grid and decides to move left twice, the ant will move to the center of the left edge then wrap around the edge and move to the center of the square. Compute the expected number of seconds it will take the ant to meet its colony (i.e. touch the right or top edge).

Problem 9

Soham the snake has laid 10 distinguishable eggs and has lined them up in a row. He is now looking to move his den elsewhere but due to his limited strength he can only take a maximum of 5 eggs with him. To ensure variability in his offspring he decides that in his subset of eggs that he takes with him, there will be no more than two pairs of eggs that are adjacent in his line of eggs. How many different combinations of eggs can Soham take with him?

Problem 10

Andrew starts off with 100 piles of stones, where the first pile has the maximum number of stones (although other piles may have as many stones as the first pile). Each second, Andrew can perform the following operation: he chooses two piles, and removes a stone from both piles.

We are given that each pile has at most 100 stones, and it is impossible for Andrew to perform operations such that 0 or 1 stones are left in the end. Let a_i be the initial size of the ith pile of stones, and let T be the number of possibilities for the ordered tuple $(a_1,...,a_{100})$. Given that T can written as $\binom{n}{k}$, where $95 < k < \frac{n}{2}$, find $n+k$.