PVMT 2023: Algebra and Number Theory Round

Problem 1

For real numbers $x > y$, we are given $(x-y)^2 = 18$ and $(x+y)^2 = 8$. Evaluate $x^2 - y^2$.

Problem 2

Woody the woodchuck is chucking wood. He chucks 1 piece of wood on the first day, 2 pieces on the second day, 4 pieces on the third day, and so on, doubling each day. If the forest has 2023 pieces of wood, how many days does it take for Woody to chuck the entire forest? If there are not enough pieces of wood, then Woody will simply chuck the rest.

Problem 3

What is the remainder when $2^{3^{4^5}}$ is divided by 7?

Note: a^{b^c} means $a^{(b^c)}$

Problem 4

We are given the two sums of infinite geometric series:

$$a + ar + ar^2 + ar^3 \ldots = 23$$
$$a - ar + ar^2 - ar^3 \ldots = 20$$

Compute the value of

$$a^2 + a^2r^2 + a^2r^4 + \ldots$$

Problem 5

There is a glass that is 2023 meters tall. I fill the bottom 1 meter with $\frac{1}{3}$ concentration lemonade (1 part lemon for 2 parts water). I fill the next $\frac{1}{2}$ meter with $\frac{1}{4}$ concentration lemonade. I continue this, on the n'th step, I fill $\frac{1}{n}$ meters with $\frac{1}{n+2}$ concentration lemonade. I do this until the glass would overflow (which will happen). Let C be the final concentration of lemonade in the glass, rounded to the nearest hundredth. Find $100C$.

Problem 6

Consider three reals $a, b, c > 0$ such that $ab^2c^3 = 2$. The cube of the minimum possible value of

$$\frac{a}{b^2c^3} + \frac{b^2}{ac^3} + \frac{c^3}{ab^2}$$

is $\frac{p}{q}$ for relatively prime positive integers p, q. What is $p+q$?

Problem 7

Let

$$f(x) = \sqrt{x^2 + 1\sqrt{x^4 + 2\sqrt{x^8 + 3\sqrt{\ldots\sqrt{x^{2^n} + n\sqrt{\ldots}}}}}}$$

Find $f^{-1}(f^{-1}(2024))$ given that $\sqrt{1 + 2\sqrt{1 + 3\sqrt{\ldots}}} = 3$.

Problem 8

Find the sum of all positive integers $n \leq 100$ for which n^2+n+31 is divisible by 43.

Problem 9

Let function $f(x)$ satisfy for $x \neq 0, -1$:

$$\frac{f(x)+1}{x+1} = \frac{f(x+1)}{x}$$

And let $\frac{1}{x} \leq f(x) \leq \frac{2}{x}$ for $0 < x < 1$. Suppose that the difference between the maximum and minimum possible values for $f(2023.6)$ can be written as $\frac{p}{q}$, for relatively prime positive integers p, q. Compute $p+q$.

Problem 10

Consider the sequence of positive reals starting with $a_2 = \frac{1}{4}$ and

$$a_n{}^{\frac{1}{n}}((n+1)!-1) = ((n+1)!-n-1)a_{n+1}{}^{\frac{1}{n+1}}$$

Find the smallest n such that $a_n > 1 - \frac{1}{2023}$.