PVMT 2022: Team Round

Problem 1

A rectangular box has a volume of 64. Find the sum of the dimensions of the box with the minimum surface area.

Solution

Let the dimensions of the box be x, y, and z. We know that $xyz = 64$, and are trying to minimize $2xy + 2yz + 2xz$. Since this function is symmetric with respect to all of the variables, the maximum occurs when $x = y = z$, so each dimension is 4 units which results in a sum of $\boxed{12}$.

Proposed by Satvik Lolla

Problem 2

Avyukth has trouble remembering his address, but knows that it is a 5 digit number such that:

1. The first two digits make a prime
2. The second and third digits make a perfect square
3. The third and fourth digits make a prime
4. The ratio of the third digit to the second is equal to the ratio of the two digit number formed by the fourth and fifth digits, to the third digit, which is equal to the first digit

What is his address?

Solution

If the first two digits form a two digit prime, then the second and third digits must be either 16 or 36, because it must start with 1, 3, 6, or 9.

The fourth digit is then either a 1 or a 7, because 61 and 67 are primes.

This leaves the possibilities of

1. -161-
2. -167-
3. -361-
4. -367-

The ratio of 1 to 6 is 6, so the last two digits would need to be 36 by our last requirement, but this is impossible because the fourth digit is known to be a 1.

We can similarly rule out 161, 176, and 367, leaving the only possibility to be -361-. The ratio of 6 to 3 is 2, so the last two digits are 12, and the first is 2, giving us the answer $\boxed{23612}$

Proposed by Milo Stammers

Problem 3

In the game of *Fim*, two players take turns removing $1, 2$, or 3 rubber falcons from the center of a table. This continues until there are no more falcons left on the table. The player that takes the last falcon from the center of the table loses.

Find the largest initial number of falcons, m, where $m \leq 100$, such that the player who moves first is guaranteed to lose. Assume perfect play from both players.

Solution

Note that if m is congruent to 1 mod 4, the first player is guaranteed to lose. For example, if there are 5 rubber falcons on the table and player 1 takes n falcons, player 2 should take $4 - n$ falcons. This ensures that the number of falcons in the center will always be congruent to 1 mod 4 and since the number of falcons is decreasing, player 1 will have to take the last falcon. The largest m congruent to 1 mod 4 and less than 100 is $\boxed{97}$.

Proposed by Orion Foo, Satvik Lolla

Comment: it is a bit more interesting (and a bit trickier) if you can remove 1, 2, or 6 falcons.

Problem 4

What is the absolute value of the sum of the 2022nd powers of the roots of this polynomial:

$$x^{2022} + 3x^2 - 2x + 1$$

Solution

For some given root to the polynomial, we have the equation:

$$r_n^{2022} + 3r_n^2 - 2r_n + 1 = 0$$

If we add this equation for all roots we get

$$S_{2022} + 3S_2 - 2S_1 + 2022 = 0$$

Where S_k denoted the sum of the k'th powers of the roots.
By Vieta's formulas, S_1=0. Also

$$S_2 = S_1^2 - 2\sum_{n}^{2022} \sum_{m=0, m\neq n}^{2022} r_n r_m$$

But the second term is just twice the coefficient of x^{2020} by Vieta's, so is equal to 0. Hence $S_2 = S_1^2 = 0$.
Thus $S_{2022} = \boxed{-2022}$

Proposed by Milo Stammers

Problem 5

If n is the number of POSITIVE integer solutions to $a+b+c+d+e=12$ and m is the number of NON-NEGATIVE integer solutions to $f+g+h=15$, find $n+m$.

Solutions

This is a classic stars and bars problem. We have to note that m accounts for the case where f, g, or $h=0$, meaning that we can place bars on the end of the list of stars as well as place multiple bars between two stars.

$$n = \binom{12-1 \text{ stars}}{5-1 \text{ bars}} = 330$$

$$m = \binom{15+3-1 \text{ stars}}{3-1 \text{ bars}} = 136$$

$$\implies n+m = \boxed{466}$$

Proposed by Orion Foo

Problem 6

Given a factorization of 2022, with no limit on the number of factors, but none of which can be one, take the sum of the factors. What is the minimum possible value?

Solution

Given a composite number mn, where neither number equals 1, it would always reduce the sum to split it into its factors m and n, or mn≥m+n.

Proof:

$$m,n \geq 2$$

$$m-1, n-1 \geq 1$$

$$(m-1)(n-1) \geq 1$$

$$mn-m-n+1 \geq 1$$

$$mn \geq m+n$$

This means that the minimum sum is achieved under the prime factorization of 2022 where no more factoring can be done, or 2*3*337, of which the sum of the factors is $\boxed{342}$.

Proposed by Milo Stammers

Problem 7

Jane and Joe are playing a game with an unfair coin. They repeatedly flip the coin until one of them wins. Jane wins if two consecutive flips are either both heads, or a tails followed by a heads. Joe wins if two consecutive flips are heads followed by tails.

Given that Jane has a 90% chance of winning, the maximum probability that the coin shows heads on a given flip can be expressed as $\frac{a}{b}+\sqrt{\frac{c}{d}}$ where a and b are relatively prime positive integers, and c and d are relatively prime positive integers. Find $a+b+c+d$.

Solution

Let p be the probability that the coin flips heads on a given flip. Note that if the coin flips tails on the first flip, Jane always wins as Joe has no way to win. If the coin flips heads on a given flip, then the next flip decides who wins: if it is heads, Jane wins, and otherwise she loses.

Thus, Jane wins with a probability of $(1-p)+p^2$ (the $1-p$ comes from flipping tails first, and the p^2 comes from flipping heads first). Setting this equal to $\frac{9}{10}$ and solving the quadratic gives that $p=\frac{1}{2}+\sqrt{\frac{3}{20}}$ so the answer is $\boxed{26}$.

Proposed by Sumedh Vangara

Problem 8

Given quadrilateral ABCD with incenter I, $\overline{AB}$=10, and $\overline{CD}$=20. What is $\overline{BC}$+$\overline{DA}$?

Solution

By Pitot's theorem, $BC+AD=AB+CD=\boxed{30}$.

Problem 9

The area of the graph enclosed by the function $|x-y|+|x+y|+x=1$ can be written as simplified fraction $\frac{p}{q}$. What is p+q?

Solution

Assuming x to be positive and have greater magnitude than y, the equation becomes

$$3x=1$$

So it makes the line x=$\frac{1}{3}$ This has boundaries at ($\frac{1}{3}$, $\frac{1}{3}$) and ($\frac{1}{3}$, -$\frac{1}{3}$), where the magnitude of y becomes greater than that of x.

Assuming x to be negative and have great magnitude than y, the equation becomes

$$-x=1$$

This makes the line x=-1, which, by a similar argument as above, has boundaries at (-1,1) and (-1,-1).

Assuming y to be positive and have greater magnitude than x, the equation becomes

$$2y+x=1$$

a simple line. And the equation has vertical symmetry because the left remains the same when we negate y. Hence the final shape is a trapezoid with bases x=-1 and x=$\frac{1}{3}$. The total area is $\frac{(2+\frac{2}{3})\frac{4}{3}}{2}$=$\frac{16}{9}$. Finally, the answer is $\boxed{25}$.

Proposed by Milo Stammers

Problem 10

How many ways are there for players to make moves to end a game of tic-tac-toe in a draw? Assume O always goes first. Rotations and reflections are distinct.

Solution

The three base configurations are shown below:

xoo xoo xoo oxx oxx oox oxo xoo xxo

Each of these have four fold rotational symmetry, but the last also can be flipped across its diagonal, so the total number of ways for the game to end is 4+4+8=16. Each of these has 4!5! ways to reach the position, because x and o can pick the order of their placements in 4! and 5! ways each. This gives the final answer of 16*24*120=$\boxed{46080}$.

Proposed by Milo Stammers

Problem 11

Take any convex equilateral hexagon(not necessarily regular) of side length 1. The angle bisector of one vertex and the bisector of the opposite vertex intersect at P. 6 altitudes are dropped from P to each side of the hexagon(or possibly their extensions). What is the maximum possible value of the sum of the lengths of these altitudes? The answer will be in the form $\sqrt{q}$, give us q.

Solution

If we were to draw lines from P to each vertex of the hexagon, we would form 6 triangles which make up to the complete hexagon. Hence is we take the sum of the areas of all the triangles, it would give us the area of the hexagon. The area of each triangle would be $\frac{1}{2}1*a$ where a is the altitude, so the sum of the areas would be $\frac{1}{2}A$ where A is the sum of the altitudes. The area of the hexagon will be maximized when it is regular(this is very difficult to prove, but it is closest to the shape of a circle, which gives maximum area for some given perimeter). The area of a regular hexagon is $\frac{3*x^2\sqrt{3}}{2}=\frac{3*1^2\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$.

$$\frac{1}{2}A=\frac{3\sqrt{3}}{2}$$

$$A=\sqrt{\boxed{27}}$$

Note: the point P is arbitrary, the answer would be the same for all points inside the hexagon.

Proposed by Milo Stammers

Problem 12

Let $f(x)$ be a function, defined as such: $f(0) = 1$, and $f(x) = 2x$ for $x \neq 0$. Let $k = f(0) + f(1) + f(2) + f(3) .... + f(256)$. What is the sum of the prime factors of k?

Solution

Note that $k = 1 + 256 \times 257 = 256^2 + 256 + 1 = 16^4 + 16^2 + 1 = (16^2 + 16 + 1)(16^2 - 16 + 1) = 273 \times 241$. Thus the prime factors of k are 3, 7, 13, 241, so the answer is 264.

Proposed by Andrew Yuan

Problem 13

Let ABC be a triangle with incenter I. Let the incircle be tangent to AB at M. Given $AM = 2 \cdot MB$, find $\frac{[AIB]}{[AIC]-[BIC]}$.

Note: $[ABC]$ denotes the area of triangle ABC.

Solution

Let $AB = c, BC = a, CA = b$ and $s = \frac{a+b+c}{2}$. Then note that $AM = s - a = 2 \cdot BM = 2 \cdot (s - b)$. Simplifying gives that $c = 3(b - a)$. Now if the inradius is r, then $[AIB] = \frac{rc}{2}, [AIC] = \frac{rb}{2}, [BIC] = \frac{ra}{2}$, and so we need to find $\frac{rc}{rb-ra} = \frac{r \cdot 3(b-a)}{r(b-a)}$ so the answer is $\boxed{3}$.

Proposed by Sumedh Vangara

Problem 14

John has an infinite deck of cards containing exactly four cards labeled each power of 7 (i.e. he has four 1's, four 7's, four 49's... etc). Let S denote the set of all distinct numbers John can obtain from adding together the labels of one or more of his cards (so for example, 7 is in S and so is $7 + 7 + 7 + 7 = 28$, but $7 + 7 + 7 + 7 + 7 = 35$ is not). Let N be the sum of the $5^{2001} - 1$ smallest elements of S. What is the remainder when N is divided by 1000?

Solution

Let $L(n)$ denote the average value of the $5^n - 1$ smallest elements in S, and $T(n)$ the sum. We then initiate an argument involving linearity of expectation to obtain the recursion $L(1) = 2$ and $L(n+1) = 7 \cdot L(n) + 2$ for all $n \geq 2$. It is easy to see that the closed form for the aforementioned recursion is $L(n) = 2\sum_{i=0}^{n-1} 7^i$. Since trivially $T(n) = 5^n \cdot L(n)$, we can write $T(2001)$ as $2 \cdot 5^{2001} \cdot \sum_{i=0}^{2000} 7^i$, which is equivalent to $125 \cdot 2 \cdot 1 = 250 \mod 1000$, so our answer is $2 + 5 + 0 = \boxed{7}$.

Proposed by Andrew Yuan

Problem 15

The complex roots of the polynomial

$$((z-(12+2i))^k-1)((z-(17-10i))^k-1)=0$$

are plotted onto the complex plane for all positive integers k. What is the smallest area of a rectangle which contains all the roots for every k?

Solution

For a number to be a root of this polynomial, then it must lie on circles radius 1 centered at 12+2i and 17-10i, making $(z-(12+2i))^k-1)$ or $((z-(17-10i))^k-1)$ zero respectively. Over all k, the roots would make a complete circle around these points. The width of the circumscribing rectangle would be $2+\sqrt{(12-17)^2+(2-(-10))^2}=15$, the +2 accounting for the radii of the circles, and the height would simply be 2, the diameter of the circles. The area of this rectangle would be $2*15=\boxed{30}$.

Proposed by Milo Stammers

Problem 16

Let ABC be a triangle inscribed in a circumcircle with radius 4 and with center O, and have incenter I. Let the tangents to the circumcircle at B and C intersect at K. If $\angle BAC=60$ and the perimeter of ABC is 16, and the product of the side lengths of ABC is 128, find KI^2.

Solution

Note that $\angle BIC=\angle BOC=120$, so B,I,O,C all lie on the same circle. Also note that $\angle KBO=\angle KCO=90$ so K,B,I,O,C are concyclic. Thus, we have $90=\angle KBO=\angle KIO$. Thus, $KI^2=KO^2-OI^2$.

To find KO, note that KCO is a $30-60-90$ triangle, so $KO=CO\cdot\sqrt{3}=4\sqrt{3}$.

To find OI, we will employ Euler's Theorem, which says $OI^2=R(R-2r)$ where R is the circumradius and r is the inradius. We already know that $R=4$, so we need to find r. To do this, we will use the formula $A=sr$ where A is the area of ABC and s is the semiperimeter. s is obviously 8. To find A, note, $A=\frac{P}{4R}$ where P is the product of the side lengths, so $A=8$, and thus $r=1$. Thus, $OI^2=4\cdot 2=8$

Thus, we have that $KI^2=48-8=\boxed{40}$.

Proposed by Sumedh Vangara

Problem 17

If the two smallest roots of $-385x^4+218x^3+144x^2+22x+1$ are a and b, then $|a+b|$ can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p+q$.

Solution

Let the polynomial in question be $P(x)$. Then

$$P(\frac{1}{x})=x^4+22x^3+144x^2+218x-385=(x-1)(x+7)(x+5)(x+11)$$

Thus, the two smallest roots are $-\frac{1}{5}$ and $-\frac{1}{7}$, and so the absolute value of their sum is $\frac{12}{35}$ and the answer is $\boxed{47}$.

Proposed by Sumedh Vangara

Problem 18

In a regular n-gon, with n ranging from 44 to 100 inclusive, each vertex is connected to its adjacent vertex and the center of the polygon. Two non-adjacent vertices are picked(can't be center), and are connected. This is done 22 times, with no vertex being picked more than once. What is the sum of all n such that it's always possible for a bug to choose a starting position, then move along the connections such that it crosses each connection exactly once?

Solution

This is a graph theory problem, we will say that each vertex and the center is a *node*, and each connection is an *edge*. We are looking for all n such that an *Euler Path* can be made, a path crossing each edge exactly once. We call the number of edges connected to some vertex its *degree*.

Lemma: An Euler path can be made if and only if there are no vertices of odd degree or there are exactly 2.

Proof:

Lets first work with the case that they are all even. Whenever the bug enters a node, it can always leave, otherwise the degree would need to be odd. The exception to this is the starting vertex, which means that it will end on this vertex(so it travels to or from it an even amount of times), forming a closed path.

If two nodes have odd degree, we can have another edge connect them so that all the degrees are even. By what we argued above, we can create an Euler Path. We need for the new edge we created to be the last. If our first movement is from one of the originally odd nodes to the other, we can still make an Euler Path by the argument above, then simply reverse it so that the first movement becomes our last.

Lastly, to show that this constitutes every Euler Path, assume there is either 1 or more than 2 odd degree nodes. If there are more than 3, then one must exist not at the beginning or end, but to enter and leave, it must have an even degree, a contradiction. If there is only 1, then it must be at the beginning or end as argued above. If the beginning is odd degree, then the end is another node, but if it never leaves the end node, then it too must have an odd degree, a contradiction. Similarly if the ending node is odd, then the starting must be odd.

This proves it is necessary and sufficient to have either no or exactly 2 odd degree nodes to create an Euler Path.

Each vertex, before adding the extra edges, is of odd degree, connects to either adjacent vertex and the center, but if it gets a connection, becomes even. The center will be of odd degree is n is odd, and even if even. If n=44, all nodes have even degree, so is a solution. If n=45, the center and one vertex are of odd degree, so is another solution. If n=46, two vertices have odd degree, so is a solution. If n>46, there are more than two odd degree vertices, so can't have a solution.

This gives us our answer, 44+45+46= $\boxed{135}$

Proposed by Milo Stammers

Problem 19

Take some number n, and define its k-ness through the process:

1. Find n mod k, if this is 0, then add 1 to its k-ness
2. Subtract n mod k from n(reducing it to the next lowest multiple of k) then divide by k
3. If n is 0 stop, otherwise repeat

Let f_k(n) denote the k-ness of n. Find

$$\sum_{m=2}^{5}\sum_{p=1}^{7-m} f_{m^p}(1798)$$

Solution

Notice that f_k(n) is simply the number of zeros in the base k representation of n. In each step, we add one if it is divisible by k, ends in a zero, then remove its modulus k and then divide by k, changing the last digit to 0 then removing it.

Notice that when we have a power p of m, m^p, then the groups of p digits in the base m representation, starting from the right, correspond to the base m^p representation when converted. This is because the maximum value reached by the group of p digits is equal to m^p. For example, a 11 in base 2 would correspond to a 3 in base 4, or 101 to 5 in base 8, because those two or three digits constitute a complete 4 or 8.

The binary representation of 1798 is: 11100000110

The base 3: 2110121

The base 5: 24143

Considering k=2^1, we are looking for the zeros in the binary representation, for which there are 6. For 2^2, we are considering pairs of 0 as 1,11,00,00,01,10 There are 2 more pairs, and we do this up to 2^5, we get 10 total. Then for 3, there is only 1. For 5 there is none. For 4, we consider even sized pairs up to 6 in length, which gives us 3 more. Our final answer is 10+1+3=$\boxed{14}$.

Proposed by Milo Stammers

Problem 20

What is the maximum value of the expression $|\frac{a-b}{5-\bar{a}b}|$, where a and b are complex numbers such that $|a| \leq 1, |b| \leq 1$. Given that the answer be written as $\frac{p}{q}$ in simplest form, find $p+q$.

Solution

Instead consider the expression

$$|\frac{a-b}{5-\bar{a}b}|^2 = \frac{(a-b)(\bar{a}-\bar{b})}{(5-\bar{a}b)(5-a\bar{b})} = \frac{|a|^2+|b|^2-\bar{a}b-a\bar{b}}{25+|a|^2|b|^2-5(\bar{a}b+a\bar{b})}$$

$\bar{a}b+a\bar{b}$ can be simplified through their exponential representations:

$$\bar{a}b+a\bar{b} = r_ar_be^{-\theta_a+\theta_b}+r_ar_be^{\theta_a-\theta_b} = |a||b|[e^{-\theta_a+\theta_b}+e^{\theta_a-\theta_b}] = 2|a||b|\cos\theta$$

Where θ is the angle between a and b

Our expression can then be written as

$$\frac{|a|^2+|b|^2-2|a||b|cos\theta}{25+|a|^2|b|^2-10|a||b|\cos\theta} = \frac{1}{5}(1-\frac{25+|a|^2|b|^2-5|a|^2-5|b|^2}{25+|a|^2|b|^2-10|a||b|\cos\theta})$$

Because the numerator and denominator are both always positive, and we want to now minimize our new fraction, it will always be optimal to let $cos(\theta)$ be -1, as it can be varied independently from the magnitudes of a and b. Next, we can factor our numerator and denominator:

$$\frac{1}{5}(1-\frac{(5-|a|^2)(5-|b|^2)}{(|a||b|+5)^2})$$

The numerator is minimized when $|a|$ and $|b|$ are both equal to 1, and the denominator is maximized with the same magnitudes. This means

$$\frac{1}{5}(1-\frac{(5-|a|^2)(5-|b|^2)}{(|a||b|+5)^2}) \leq \frac{1}{5}(1-\frac{16}{36}) = \frac{1}{9}$$

Finally, we want to take the square root of our answer because this is the square magnitude, and we an answer of $\frac{1}{3}$. Our final answer is $\boxed{4}$.

Proposed by Milo Stammers