PVMT 2022: Middle School Division Geometry Round

Problem 1

Alice is standing 5 meters away from Bob. Bob is standing 20 meters away from Charlie. In meters, what is the minimum possible distance between Alice and Charlie?

Solution

The minimum distance is achieved when the three people are in line, Alice between Bob and Charlie. Their distance would then be $20-5=\boxed{15}$.

Proposed by Andrew Yuan

Problem 2

Let S be the set of points in the coordinate plane that are a distance 11 away from the origin. Let T be the set of points that are a distance of at most 3 from a point in S. If the area of T can be expressed as $n\pi$, find n.

Solution

Note that T contains all the points that are over a distance of $11-3=8$ from the origin, and less than $11+3=14$ from the origin. This implies that T is a circle of radius 14 with a circle of 8 "cutöut of it. Thus, the desired area is

$$14^2\pi - 8^2\pi = 132\pi$$

so the answer is $\boxed{132}$.

Proposed by Sumedh Vangara

Problem 3

In $\triangle ABC$, $AB=5$, $BC=12$, and $AC=13$. A square with side length s is constructed such that one of its sides lies on AC, and each of its vertices lies on a side of $\triangle ABC$. Given that s can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

Solution

Let D be the corner of the square that lies on side AB. By similar triangles, we have $BD=\frac{5}{13}$ and $AD=\frac{13}{12}s$, so $\frac{5}{13}s+\frac{13}{12}s=\frac{229}{156}s=5$. Thus, $s=\frac{780}{229}$, and our answer is $\boxed{1009}$.

Proposed by Andrew Yuan

Problem 4

Take the ellipse with equation

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

It is then rotated 30 degrees clockwise and a rectangle is circumscribed around it. What is the minimum possible area of this rectangle?

Solution

The rotation can be ignored, it won't change the area of the rectangle. Notice here that we can dilate the ellipse and rectangle by a factor of $\frac{1}{3}$ along the x-axis and a factor of $\frac{1}{4}$ along the y-axis to create a circle of radius 1 with a square circumscribing it. The area of the rectangle will now be $1/12$ of what it was originally. The square will always have a side length of 2, so its area 4. To dilate back would give us the area of the rectangle, 4*12=$\boxed{48}$.

Proposed by Milo Stammers

Problem 5

What is the volume of the region enclosed by both the graphs

$$x^{2022}+y^{2022}+z^{2022}=1,(x-\frac{1}{2})^{2022}+y^{2022}+(z+1)^{2022}=2$$

Round your answer to the nearest integer.

Solution

Notice that the boundary of the first equation will be just inside the planes x=1, y=1, and z=1. It is approximately a cube. Similarly with the second equation, it bulges out slightly at the faces, and slightly inwards by the corners, but again is effectively a cube.

The volume enclosed would then be a prism with side lengths 2, $\frac{3}{2}$, and 1. Thus the volume is $\boxed{3}$.

Proposed by Milo Stammers

Problem 6

Points A, B, C, and D are in a Cartesian plane such that $A=(2048,2058)$, $B=(2018,2018)$, $C=(2036,2046)$, and $D=(2060,2039)$. Given that point P is in the same plane, the minimum possible value of $PA+PB+PC+PD$ is closest to which integer?

Solution

Plotting the points, we see that the points form a convex quadrilateral $ACBD$. The value of $PA+PB+PC+PD$ is minimized when P is the intersection of the diagonals of the quadrilateral (this can be proven with triangle inequality), in which case that value will be equal to the sum of the diagonals, which is $AB+CD$. Thus, our answer is $AB+CD=50+25=\boxed{75}$.

Proposed by Andrew Yuan

Problem 7

Let $\triangle ABC$ be an equilateral triangle, and let P be a point inside $\triangle ABC$ such that the distances from P to AB, AC, and BC are 2, 4, and 7, respectively. The area of $\triangle ABC$ can be written as $\frac{m}{\sqrt{n}}$, where m is a positive integer and n is not divisible by the square of any prime. What is $m+n$?

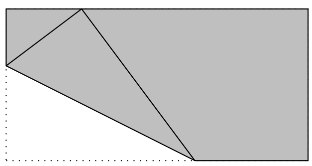
Solution

Let x be the side length of the equilateral triangle. Due to the distances from P, the area of the triangle is $\frac{2x+4x+7x}{2} = \frac{13}{2}x$. So $\frac{13}{2}x = \frac{\sqrt{3}x^2}{4}$, and $x = \frac{26}{\sqrt{3}}$. So $[ABC] = \frac{13}{2}x = \frac{169}{\sqrt{3}}$, and our answer is $\boxed{172}$.

Proposed by Andrew Yuan

Problem 8

A 2 by 1 strip of paper is folded so that a corner meets a point a distance x along on the opposite edge (see figure). Let x be the value such that the area of the resulting shape is maximized. Suppose x^4 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Solution

Let the top left, top right, bottom right, and bottom left corners (as shown in the diagram) of the strip of paper be A, B, C, D respectively. Let E be the intersection of the crease (from the fold) with AD, and let F be the intersection of the crease with CD. Let G be the point where the corner meets AB once the paper is folded. Finally, let $DE = GE = y$ and $AG = x$; we have $AE = 1-y$.

Now, note that by the perpendicular from F to AB has length 1, so by similar triangles, $FG = \frac{y}{x}$, and $[EFG] = \frac{y^2}{2x}$. To maximize the shaded area we want to minimize $[DEF] = [EFG]$, so we just want to minimize $\frac{y^2}{x}$. But note $x^2 = y^2 - (y-1)^2 = 2y-1$, so $y = \frac{x^2+1}{2}$, and $\frac{y^2}{x} = \frac{(x^2+1)^2}{4x} = \frac{x^3}{4} + \frac{x}{2} + \frac{1}{4x}$. So we just want to minimize this expression.

Taking the first derivative of the expression, we get $\frac{3x^2}{4} + \frac{1}{2} - \frac{1}{4x^2}$. The optimal x must be a critical point, so we just solve $\frac{3x^2}{4} + \frac{1}{2} - \frac{1}{4x^2} = 0$; this can be rewritten as $3x^4 + 2x^2 - 1 = 0$, or $(3x^2-1)(x^2+1) = 0$. The only positive x that satisfies this is $x = \sqrt{\frac{1}{3}}$. We can check that this indeed works by taking the second derivative of the function and seeing that it's convex for all $x > 0$. So thus, $x^4 = \frac{1}{9}$, and our answer is $\boxed{10}$.

Proposed by Caleb Dastrup, solution by Andrew Yuan

Problem 9

A point P is placed in equilateral triangle $\triangle ABC$ such that $\angle APB = 100$, $\angle BPC = 120$, and $\angle APC = 140$. A triangle $\triangle XYZ$ is constructed such that $AP = XY$, $BP = XZ$, and $CP = YZ$. Find the measure of the largest angle of $\triangle XYZ$.

Solution

Rotate $\triangle APC$ about C such that A' coincides with B. Note that $CP = CP'$ and $\angle PCP' = 60$, so $\triangle PCP'$ is equilateral and $CPP' = CP'P = 60$. Now, note that $\angle PP'B = 140 - 60 = 80$, $\angle P'PB = 120 - 60 = 60$, and $\angle PBP' = 180 - 140 = 40$. Also note that $CP = PP'$ and $AP = BP'$, meaning that $\triangle PBP' \cong \triangle XYZ$. It follows that the answer is $\boxed{80}$.

Proposed by Andrew Yuan

Problem 10

In triangle $\triangle ABC$, we have $AB = 13$, $BC = 14$, and $AC = 15$. Let I be the incircle of $\triangle ABC$, and let D, E, F be the tangency points of the incircle to sides BC, AC, AB, respectively. Let X be the foot of the perpendicular from D to EF, let Y be the midpoint of DX, and let K be the orthocenter of $\triangle BIC$. Finally, let Z be the intersection between YK and EF. The value of $(ZE)(ZF)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution

The key claim in our problem is that Z is the midpoint of EF. To prove this, we will first establish two lemmas.

Lemma 1: In arbitrary $\triangle ABC$, let the incenter be I, and let the contact triangle be $\triangle DEF$. Then, the C-midline, line EF, line BI, and altitude from C to BI concur at J. (Similarly for other vetex/chord combinations.)

Proof: Redefine $J = EF \cap BI$. Note that $\angle EJI = \angle FJI = 180 - \angle B/2 - (90 + \angle A/2) = \angle C/2$, meaning $EJCI$ is cyclic. Thus, since $\angle IEC = 90$, we have $\angle IJC = 90$ as well. Now let M be the midpoint of BC and N be the midpoint of AC. Note that since $\triangle JBC$ is right, we have $JM = BM = CM$, so $\angle JMB = 180 - 2 \cdot (\angle B/2) = 180 - \angle B$, meaning J, M, N are collinear. End lemma.

Lemma 2: In arbitrary $\triangle ABC$, let M be the midpoint of the altitude from A to BC. Also, let D be the tangency point between the incircle and BC, I be the incenter, I_A be the A-excenter, X be the tangency point between the A-excircle and BC, and Y be the point on the A-excircle diametrically opposite of Y. Then, M, I, X are collinear, and M, D, Y are collinear.

Proof: Let E be the point on the incircle diametrically opposite of D, and let K be the foot from A to BC. Due to the homothety about A between the incircle and excircle, we have that A, E, X are collinear, and A, D, Y are collinear. Thus, $\triangle AXK, \triangle EXD$ are homothetic about X so M, I, X are collinear, and $\triangle ADK, \triangle YDX$ are homothetic about D so M, D, I_A are collinear. End lemma.

Now we put together the two lemmas to solve our original problem. It is well known that AI is the perpendicular bisector of EF, so let $N = EF \cap AI$, and it suffices to prove Y, N, K collinear. Now, let Q be foot from C to line BI and R be the foot from B to line CI, and note that by Lemma 1, we have that Q, R are both on line EF. Now, note that $\triangle DQR$ is the orthic triangle of $\triangle BKC$, so K is the D-excenter of $\triangle DQR$. But by Lemma 2 on $\triangle DQR$, we see that Y, N, K are collinear, as desired.

Therefore, we have that YK meets EF at N, meaning $N = Z$, and thus Z is the midpoint of EF. The rest is trivial computation: Heron's formula gives us $[ABC] = 84$, so since the semiperimeter is $s = \frac{13+14+15}{2} = 21$, the inradius is $r = IF = IE = 4$. Also, using the fact that $AE = AF$, $BF = BD$, and $CE = CD$, it's easy to calculate that $AE = AF = 7$. Thus, $AI = \sqrt{65}$, and $EZ = FZ = \frac{4 \cdot 7}{sqrt 65} = \frac{28}{\sqrt{65}}$, meaning $(ZE)(ZF) = \frac{784}{65}$. This gives us a final answer of $\boxed{849}$.

Proposed by Andrew Yuan