PVMT 2022: Middle School Division Combinatorics Round

Problem 1

How many ways are there to arrange 5 different people in a line?

Solution

The answer is simply $5! = \boxed{120}$.

Proposed by Andrew Yuan

Problem 2

In a bag, there are two green marbles and five blue marbles. Bob chooses two marbles from the bag at random, without replacement. If the probability that he chooses two marbles of different colors is $\frac{p}{q}$ with $gcd(p,q) = 1$, find $p+q$.

Solution

The probability is $\frac{2}{7} \times \frac{5}{6} \times 2 = \frac{10}{21}$, so the answer is $\boxed{31}$.

Proposed by Andrew Yuan

Problem 3

Bob has a jar that can hold up to one gallon of water. Bob first pours out half the water from the jar, and then pours out a third of the remaining water, and then pours out a fourth of the remaining water... and so on. Starting from a full jar, after how many pours will Bob have $\frac{1}{2022}$ gallons of water left?

Solution

On the first pour you multiply the water in the jar by $\frac{1}{2}$, on the second pour by $\frac{2}{3}$, ..., on the nth pour by $\frac{n}{n+1}$. So after n pours, there will be $(\frac{1}{2})(\frac{2}{3})(\frac{n}{n+1}) = \frac{1}{n+1}$ gallons of water left. In this case, $n = \boxed{2021}$.

Proposed by Andrew Yuan

Problem 4

Franklin is stuck inside of the world of Kemsuk! He is currently residing on the origin and in order to escape, he must cross a 6×8 grid and reach the land of Fisbest located at the coordinates $(6,8)$.

At any point, Franklin can only run 1 unit up or 1 unit to the right. How many paths are there from Kemsuk to Fisbest?

Solution

Franklin is only permitted to run up or run to the right so he makes a total of $8+6=14$ moves. In order to reach Fisbest, he must choose to run up exactly 8 times and the rest of his moves must subsequently be moves to the right (or 6 right, and the rest up).

This gives us:

$$\binom{14}{8} = \binom{14}{6} = \boxed{3003 \text{ paths}}$$

Proposed by Orion Foo

Problem 5

In a lottery, a ticket consists of 5 distinct, unordered digits between 0 and 9. What is the minimum number of lottery tickets that needs to be created such that it can be guaranteed that at least 6 lottery tickets are identical?

Solution

This problem can be solved using the Pigeonhole Principle, where the ways in which a ticket can be seen as a hole and each ticket can be seen as a pigeon. In order to guarantee that at least 6 lottery tickets are identical, we would need $5n+1$ tickets, where n is the number of ways that a ticket can be made. With $5n+1$ tickets, there is enough tickets such that, in the worst case scenario, there are $n-1$ sets of 5 identical tickets and 1 set of 6 identical tickets. In this problem, the number of ways a ticket can be made is given as

$$\binom{10}{5} = 252,$$

$$252 * 5 + 1 = \boxed{1261}$$

Proposed by Jeffrey Jiang

Problem 6

You flip a coin and roll a die, if the number on the die is odd and the coin lands tails or the number on the die is even and the coin lands heads, then you stop, otherwise you repeat. What is the expected number of times you do this before you stop?

Solution

The probability of getting an odd is $\frac{1}{2}$ and the probability of landing tails is also $\frac{1}{2}$, therefore the probability of getting both is $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$. Similarly, the probability of getting an even and heads if $\frac{1}{4}$. Thus, the probability of getting either would be $\frac{1}{2}$. Intuitively, we get that the expected number of turns would be $\boxed{2}$.

This can be more rigorously seen through expectation value, the product of probability of stopping on the nth turn times the n turns. The probability of stopping on turn 1 is $\frac{1}{2}$, on the second would be $\frac{1}{4}$, and so on, dividing by two each time. The expectation value is:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}...=$$

$$[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}...]+$$

$$[\frac{1}{4}+\frac{1}{8}+\frac{1}{16}...]+$$

$$[\frac{1}{8}+\frac{1}{16}...]+$$

$$...$$

$$=1+\frac{1}{2}+\frac{1}{4}...\boxed{2}$$

From the geometric series formula.

Proposed by Milo Stammers

Problem 7

Oryan, a student at Fisbad, throws 2 balls into containers labeled $1,2,...,n,...$ (there are infinitely many total containers). If the probability that each ball lands in container k is 2^{-k}, the probability that the balls land in the same container can be represented as $\frac{p}{q}$ where $\gcd(p,q)=1$ Find $p+q$.

Solution

The probability that both balls land in container k is simply 2^{-2k}, or $\frac{1}{4^k}$. So summing this up for all k, we get $\frac{1}{4}+\frac{1}{16}+...=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}$, so the answer is $\boxed{4}$.

Proposed by Satvik Lolla

Problem 8

At a school:

1. Students can take either French or Spanish
2. Students cannot take both French and Spanish, but can take neither
3. 250 students take French
4. 300 students take Spanish
5. 300 students take history
6. Every student takes at least one of : French, History, and Spanish

What is the difference between the maximum and minimum number of students that could be at the school?

Solution

Let 0≤x≤250 be the number of students taking only French of the three classes, and 0≤y≤300 be the number taking only Spanish. The number of students taking French and history would then be 250-x and the number taking Spanish and history 300-y. Hence the number taking only history will be 300-(250-x)-(300-y)=x+y-250. Therefore x+y≥250.

The total number of students in the school will be 250+300+x+y-250=300+x+y. The minimum possible value of this is 300+250=550 because x+y≥250, and the max would be 300+250+300=850, because x≤250 and y≤300. Finally, the difference would be 850-550=$\boxed{300}$.

Proposed by Milo Stammers

Problem 9

A bug starts at $(0,0)$ on a Cartesian plane. Each move, the bug can move to any lattice point exactly $\sqrt{5}$ units away (essentially like a knight in chess), as long as it stays inside or on the square with vertices $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$. Let A be the probability that after 300 moves, the bug is back on $(0,0)$; also, let B be the probability that after 300 moves, the bug is on $(2,2)$. The value of $A-B$ can be written as $\frac{1}{n}$ for a positive integer n. Find the remainder when n is divided by 100.

Solution

Label $(0,0)$ as A, label $(0,1)$ and $(1,0)$ as B, label $(0,2)$ and $(2,0)$ as CC, label $(1,2)$ and $(2,1)$ as D, and label $(2,2)$ as E. Let A_n be the probability of landing on a point labeled A after n moves, B_n be the probability of landing on a point labeled B after n moves, etc.

Now we use recursion to compute A_n. Look two moves back: to get to A on the nth move, we either go from A to D to A (i.e. a point labeled A to a point labeled D to a point labeled A) or from C to D to A. The former has probability $A_{n-2}\cdot 1\cdot \frac{1}{2}$, and the latter $C_{n-2}\cdot\frac{1}{2}\cdot\frac{1}{2}$. Thus, $A_n=\frac{1}{2}A_{n-2}+\frac{1}{4}C_{n-2}$. Similarly we compute E_n: to get to E on the nth move, we either go from E to B to E or C to B to E. The former has probability $\frac{1}{2}E_{n-2}$, and the latter $\frac{1}{4}C_{n-2}$. Thus, $E_n=\frac{1}{2}E_{n-2}+\frac{1}{4}C_{n-2}$.

Subtracting, we have $A_n-E_n=\frac{1}{2}(A_{n-2}-E_{n-2})$. Note that when $n=2$, we have $A_2=\frac{1}{2}$ and $E_2=0$, so $A_2-E_2=\frac{1}{2}$. Thus, $A_{300}-E_{300}=\frac{1}{2^{150}}$, so our answer is $2^{150} \pmod{100}=2^{10} \pmod{100}=\boxed{24}$.

Proposed by Andrew Yuan

Problem 10

In the equation $x=4?4?4?4?4?4?4?4?4$, each question mark is randomly replaced by one of the four operators ($+$, $-$, $\times$ or $\div$). If standard order of operations is followed, then the expected value of x can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the number of positive factors of n.

Solution

First of all, note that there are nine 3's and eight blank spaces for the operators. We split this problem into two cases:

1.) There are no $+$'s and $-$'s. First note that the first 3 is fixed as $\times 3$, and for each of the other eight 3's, it can be either $\times 3$ or $\times \frac{1}{3}$. Hence, the sum of all possible values for this case is $4(4+\frac{1}{4})^8 = 4(\frac{17}{4})^8 = \frac{17^8}{4^7}$. Since each of these combinations has a $\frac{1}{4^8}$ chance of being chosen (since there are 8 spaces), case 1 contributes a total $\frac{17^8}{4^7} \cdot \frac{1}{4^8} = \frac{17^8}{4^{15}}$ to the expected value.

2.) There is at least one $+$ or $-$ Note that each expression formed in this way will have a "conjugate"(for example, $4 \times 4 \div 4 + 4 - 4 \times 4$ has a "conjugateöf $4 \times 4 \div 4 - 4 + 4 \times 4$), so this will make everything to the right of the first (leftmost) $+$ or $-$ cancel out. Hence, we only need to compute the sum of all possible values of the 4's to the left of the first $+$ or $-$. Now, suppose that the first $+$ or $-$ is the nth operator (where $n \in [1,8]$). The expected value of the expression left of this operator is $4(4+\frac{1}{4})^{n-1} = \frac{17^{n-1}}{4^{n-2}}$, and the probability of this happening is $\frac{1}{2^n}$. Thus, this n contributes $\frac{17^{n-1}}{4^{n-2}} \cdot \frac{1}{2^n}$ to the expected value, so in total case 2 contributes $\sum_{n=1}^{8} \frac{17^{n-1}}{4^{n-2}} \cdot \frac{1}{2^n}$.

Therefore, the expected value of x is $\frac{17^8}{4^{15}} + \sum_{n=1}^{8} \frac{17^{n-1}}{4^{n-2}} \cdot \frac{1}{2^n} = \frac{17^8}{2^{30}} + \sum_{n=1}^{8} \frac{17^{n-1}}{2^{3n-4}}$. The denominator of the second term is at most 2^{20}, so the denominator of the sum is just the denominator of the first fraction, which is 2^{30}. Thus the answer is $\boxed{31}$.

Proposed by Andrew Yuan