PVMT 2022: High School Division Combinatorics Round

Problem 1

How many ways are there to arrange 5 different people in a line?

Solution

The answer is simply $5! = \boxed{120}$.

Proposed by Andrew Yuan

Problem 2

Franklin is stuck inside of the world of Kemsuk! He is currently residing on the origin and in order to escape, he must cross a 6×8 grid and reach the land of Fisbest located at the coordinates $(6, 8)$.

At any point, Franklin can only run 1 unit up or 1 unit to the right. How many paths are there from Kemsuk to Fisbest?

Solution

Franklin is only permitted to run up or run to the right so he makes a total of $8 + 6 = 14$ moves. In order to reach Fisbest, he must choose to run up exactly 8 times and the rest of his moves must subsequently be moves to the right (or 6 right, and the rest up).

This gives us:

$$\binom{14}{8} = \binom{14}{6} = \boxed{3003 \text{ paths}}$$

Proposed by Orion Foo

Problem 3

Mr. Gerard is playing a betting game with his favorite students during lunch, with weighted coin that lands on heads 20% of the time. In the game, the weighted coin is flipped n times and Mr. Gerard bets that the coin will land on heads exactly 3 times. Given that the probability of the coin landing on heads exactly 3 times in n flips is the same as the probability that the coin lands on heads exactly 4 times in n flips, find the value of n.

Solution

The probability that the coin lands on heads exactly 3 times is given by:

$$\binom{n}{3} \cdot (\frac{1}{5})^3 \cdot (\frac{4}{5})^{n-3}$$

. Similarly, the probability that the coin lands on heads exactly 4 times is given by:

$$\binom{n}{4} \cdot (\frac{1}{5})^4 \cdot (\frac{4}{5})^{n-4}$$

. Equating these two expressions gives us,

$$\binom{n}{3}\cdot(\frac{1}{5})^3\cdot(\frac{4}{5})^{n-3}=\binom{n}{4}\cdot(\frac{1}{5})^4\cdot(\frac{4}{5})^{n-4}$$

$$(\frac{1}{5})^{-1}\cdot\frac{4}{5}=\frac{\binom{n}{4}}{\binom{n}{3}}$$

$$4=\frac{\frac{n!}{4!(n-4)!}}{\frac{n!}{3!(n-3)!}}$$

$$4=\frac{n-3}{4}$$

$$\implies n=\boxed{19}.$$

Proposed by Orion Foo

Problem 4

A city is holding a lottery. In this lottery, a person can pick tickets that consist of 5 unique digits between 0 and 9. When the winning ticket is drawn, the holder of a ticket that has all 5 of winning digits gets \$1036 (the order of the digits doesn't matter). However, the lottery also includes smaller prizes for when 3 or 4 digits match. When a ticket has 4 matching digits, the holder wins \$100, and when a ticket has 3 matching digits, the holder wins \$10.

What is the minimum value that the cost of the tickets need to be greater than such that the city can be expected to profit off of the lottery?

Solution

In order for the lottery to be expected to be profitable for the city, the expected value of a ticket needs to be less than the cost of a ticket. The expected value of a ticket can be represented as:

$$E[X]=\sum_{i=1}^{n}x_i * p_i,$$

where $E[X]$ is the expected value of a ticket, x_i is a possible value or outcome of a ticket, and p_i is the probability associated with said outcome. In this problem, there are 3 non-zero outcomes, \$1036 when 5 digits match, \$100 when 4 digits match, and \$10 when 3 digits match. To obtain the probabilities associated with these outcomes, we need to find the number of ways in which a ticket could be made to qualify for the outcome and divide it by the total number of ways in which a ticket could be made. This can be found using the following expression:

$$\frac{\binom{M}{r}\binom{N-M}{M-r}}{\binom{N}{M}},$$

where M is the number of digits in a ticket, N is the number of digits that can be chosen from, and r is the number of matching digits. From that, $\binom{M}{r}$ represents the number of ways the correct digits can be chosen, $\binom{N-M}{M-r}$ represents the number of ways the incorrect digits can be chosen, and $\binom{N}{M}$ represents the total number of ways ticket can be chosen.

From this, the expected value of a ticket can be calculated as:

$$E[X] = 1036\frac{\binom{5}{5}\binom{5}{0}}{\binom{10}{5}} + 100\frac{\binom{5}{4}\binom{5}{1}}{\binom{10}{5}} + 10\frac{\binom{5}{3}\binom{5}{2}}{\binom{10}{5}}$$
$$= \frac{1036+2500+1000}{252}$$
$$= \boxed{18}$$

Proposed by Jeffrey Jiang

Problem 5

Aidan has 2022 boxes numbered from 1 to 2022 and an infinite number of balls. He can throw a ball into any box which is a prime number less than 20: 2, 3, 5, 7, ..., 19. If three balls occupy the same box, then they are all removed and a new ball is put into the next. Aidan wants a ball in the 2022nd box. Let T be the ratio of the maximum possible number of throws to the minimum possible number of throws that accomplishes this. Given that T can be expressed as p^q for positive integers p, q, find pq.

Solution

To fill them as fast as possible, he would throw each ball into box 19, or box 2 to go as slow as possible. The ratio of throws would be how many times you would need to throw a ball into box 2 to get one into box 19. Three balls would correspond to one in box 3, nine would correspond to one in box 4, and this grows exponentially with base 3. Thus the ratio would be $3^{19-2} = 3^{17}$, giving us our answer of $3 \times 17 = \boxed{51}$.

Proposed by Milo Stammers

Problem 6

You are playing a game with a friend. Your probability of winning one round is $\frac{1}{10}$. But your friend is nice and says that if you lose the first round, you can play twice more, and if you win the second and third, you win the entire game. In fact, if you lose the first 2 games, then you can play three more, and if you win all three of those rounds, you win the game. This rule applies for any first n rounds you lose, if you win the next n+1 you win the entire game. What is the probability of you winning? Your answer will be a fraction written simplified as $\frac{p}{q}$, give us p+q.

Solution

The probability of you winning off the bat is $\frac{1}{10}$. Say you lose the first game, then win the next two, the probability of that is

$$(\frac{9}{10})(\frac{1}{10})^2$$

The probability of you losing the first n games then winning the next n+1 is

$$(\frac{9}{10})^n * (\frac{1}{10})^{n+1}$$

Summing this from n=0, you winning the first game, to infinity, we get

$$\sum_{n=0}^{\infty}(\frac{9}{10})^n * (\frac{1}{10})^{n+1} = \frac{\frac{1}{10}}{1-\frac{9}{10}*\frac{1}{10}} = \frac{10}{91}$$

Hence p+q=$\boxed{101}$.

Proposed by Milo Stammers

Problem 7

A new gacha game, *Shingen Pactim*, just came out and it's all the rage. In Shingen Pactim, each destinyä player spends allows them a chance at obtaining a featured 5* character. The chance is normally 20%, but starting from the 75th destiny spent, the chance increases linearly until it hits 100% for the 90th roll. Timmy really wants the newest 5* character to round out his team composition, but he doesn't have enough destinies for 90 rolls. He's already spent 73 destinies with no luck and only has 3 destinies left.

If the probability that Timmy will get the character within those 3 wishes can be expressed as $\frac{m}{n}$ where m and n are two relatively prime whole numbers, what is $m+n$?

Solution

Let's tackle this problem with casework. Since the 5* rate increases linearly starting from the 75th roll until it hits 100% on the 90th roll, then the probability of pulling a 5* increases by 5% every destiny spent.

We must consider the following cases:

Case 1: Timmy rolls for a 5* on his first roll (the 74th destiny). This gives us a probability of $\frac{1}{5}$.

Case 2: Timmy gets a 5* on his second roll (the 75th destiny). Probability of $\frac{4}{5} \cdot \frac{20+5}{100} = \frac{1}{5}$.

Case 3: Similarly for rolling a 5* on his third roll:
Probability of $\frac{4}{5} \cdot \frac{75}{100} \cdot \frac{20+5+5}{100} = \frac{9}{50}$
Simply adding our cases together, we get: $\frac{29}{50} \implies \boxed{79}$.

Proposed by Orion Foo

Problem 8

On some *m*x*m* grid of squares, an n-L is a move where, from your starting square, move n squares in one direction, then another 1 to the left or right in that direction of motion. This is an elongated L move a knight makes in chess.

Let f(n) be the least m such that for all M≥m, starting in a corner, you can reach every square of an MxM grid using only n-L moves. If there is no such value, f(n)=0.

What is f(2022)?

Solution

Note that if n is odd, then f(n)=0. If were to alternate each square black and white, then you can never leave the starting color because you make some odd number of moves, followed by one more, a total even number of moves.

If n is even, then consider a (n+1)x(n+1) grid of squares. Starting in the corner, we can reach every point on the outer ring of this grid. Starting on the top left, go all the way to the right and down one, then to the left and down another, and repeat this process, and because of parity, you will arrive at the bottom left corner. Repeat this process from top to bottom to reach the bottom right, then left and right again to the top right, then top and bottom back to the start position. With this, you have reached every point on the ring.

Using this ring on a larger grid, we can reach all those squares, and can shiftthe ring. To shift right, we start at the top left, then perform a sequence of moves to go right one, which can be done because it is contained within our ring. From here we can created a new ring with this as our top left corner. You can similarly shift in any direction.

Using this ring shifting process, we can complete an entire grid so long as there are no center points that cannot be reached. The smallest grid would be 2(n+1)-2=2n. This is because if we take the center four squares, we can create a ring with one corner in the center, and the corner diagonally across in the opposite corner of the entire grid. For example, a ring would use the top left center grid space and the bottom right of the entire board, this would have length 2n+1, our ring size. If it were any smaller, there would be a grid space in the center that cannot be reached with any ring.

Finally, we get our answer 2022*2= $\boxed{4044}$.

Proposed by Milo Stammers

Problem 9

A bug starts at $(0,0)$ on a Cartesian plane. Each move, the bug can move to any lattice point exactly $\sqrt{5}$ units away (essentially like a knight in chess), as long as it stays inside or on the square with vertices $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$. Let A be the probability that after 300 moves, the bug is back on $(0,0)$; also, let B be the probability that after 300 moves, the bug is on $(2,2)$. The value of $A-B$ can be written as $\frac{1}{n}$ for a positive integer n. Find the remainder when n is divided by 100.

Solution

Label $(0,0)$ as A, label $(0,1)$ and $(1,0)$ as B, label $(0,2)$ and $(2,0)$ as CC, label $(1,2)$ and $(2,1)$ as D, and label $(2,2)$ as E. Let A_n be the probability of landing on a point labeled A after n moves, B_n be the probability of landing on a point labeled B after n moves, etc.

Now we use recursion to compute A_n. Look two moves back: to get to A on the nth move, we either go from A to D to A (i.e. a point labeled A to a point labeled D to a point labeled A) or from C to D to A. The former has probability $A_{n-2} \cdot 1 \cdot \frac{1}{2}$, and the latter $C_{n-2} \cdot \frac{1}{2} \cdot \frac{1}{2}$. Thus, $A_n = \frac{1}{2}A_{n-2} + \frac{1}{4}C_{n-2}$. Similarly we compute E_n: to get to E on the nth move, we either go from E to B to E or C to B to E. The former has probability $\frac{1}{2}E_{n-2}$, and the latter $\frac{1}{4}C_{n-2}$. Thus, $E_n = \frac{1}{2}E_{n-2} + \frac{1}{4}C_{n-2}$.

Subtracting, we have $A_n - E_n = \frac{1}{2}(A_{n-2} - E_{n-2})$. Note that when $n = 2$, we have $A_2 = \frac{1}{2}$ and $E_2 = 0$, so $A_2 - E_2 = \frac{1}{2}$. Thus, $A_{300} - E_{300} = \frac{1}{2^{150}}$, so our answer is $2^{150} \pmod{100} = 2^{10} \pmod{100} = \boxed{24}$.

Proposed by Andrew Yuan

Problem 10

In the equation $x = 4?4?4?4?4?4?4?4?4$, each question mark is replaced by one of the four operators ($+$, $-$, $\times$ or $\div$). If standard order of operations is followed, then the expected value of x can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the number of positive factors of n.

Solution

First of all, note that there are nine 3's and eight blank spaces for the operators. We split this problem into two cases:

1.) There are no $+$'s and $-$'s. First note that the first 3 is fixed as $\times 3$, and for each of the other eight 3's, it can be either $\times 3$ or $\times\frac{1}{3}$. Hence, the sum of all possible values for this case is $4(4+\frac{1}{4})^8 = 4(\frac{17}{4})^8 = \frac{17^8}{4^7}$. Since each of these combinations has a $\frac{1}{4^8}$ chance of being chosen (since there are 8 spaces), case 1 contributes a total $\frac{17^8}{4^7} \cdot \frac{1}{4^8} = \frac{17^8}{4^{15}}$ to the expected value.

2.) There is at least one $+$ or $-$ Note that each expression formed in this way will have a "conjugate"(for example, $4 \times 4 \div 4 + 4 - 4 \times 4$ has a "conjugateöf $4 \times 4 \div 4 - 4 + 4 \times 4$), so this will make everything to the right of the first (leftmost) $+$ or $-$ cancel out. Hence, we only need to compute the sum of all possible values of the 4's to the left of the first $+$ or $-$. Now, suppose that the first $+$ or $-$ is the nth operator (where $n \in [1,8]$). The expected value of the expression left of this operator is $4(4+\frac{1}{4})^{n-1} = \frac{17^{n-1}}{4^{n-2}}$, and the probability of this happening is $\frac{1}{2^n}$. Thus, this n contributes $\frac{17^{n-1}}{4^{n-2}} \cdot \frac{1}{2^n}$ to the expected value, so in total case 2 contributes $\sum_{n=1}^{8} \frac{17^{n-1}}{4^{n-2}} \cdot \frac{1}{2^n}$.

Therefore, the expected value of x is $\frac{17^8}{4^{15}} + \sum_{n=1}^{8} \frac{17^{n-1}}{4^{n-2}} \cdot \frac{1}{2^n} = \frac{17^8}{2^{30}} + \sum_{n=1}^{8} \frac{17^{n-1}}{2^{3n-4}}$. The denominator of the second term is at most 2^{20}, so the denominator of the sum is just the denominator of the first fraction, which is 2^{30}. Thus the answer is $\boxed{31}$.

Proposed by Andrew Yuan