PVMT 2022: Middle School Division Algebra/Number Theory Round

Problem 1

Find the number of positive integers less than or equal 100 that are divisible by neither 2 nor 5.

Solution

From 1 through 100 there are 50 multiples of 2, 20 multiples of 5, and 10 multiples of 2 and 5. So the answer is simply $100-50-20+10=\boxed{40}$.

Proposed by Andrew Yuan

Problem 2

How many digits are in the number $8^5 \times 5^{10}$?

Solution

We rewrite 8^5 as 2^{15}. Our product now simplifies to $10^{10} * 2^5$. This number is equal to $3.2 * 10^{11}$, so it has $\boxed{12}$ digits.

Proposed by Orion Foo

Problem 3

In the land of Kemgod, Franklin can use each dollar to buy either 5 grams of chemical A or 6 grams of chemical B. Let a number n be *purchasable* if Franklin can spend some amount of money to buy n total grams of chemicals. If Franklin has infinite money, what is the largest number that is not *purchasable*?

Solution

This follows from the postage-stamp theorem, which states that the largest number that is not purchasable is equal to $5\times 6-5-6=\boxed{19}$.

Proposed by Satvik Lolla

Problem 4

Consider the polynomial $f(x)=x^2+4x-12$. Find the absolute value of the sum of all x such that $f(f(x))=0$.

Solution

Completing the square gives $f(x)=(x+2)^2-16$. $f(f(x))$ can then be written as $((x+2)^2-16+2)^2-16$. Setting this equal to 0, we have the following:

$$\begin{aligned}((x+2)^2-16+2)^2-16&=0\\(x+2)^2-14&=\pm 4\end{aligned}$$

This gives two equations: $(x+2)^2 = 18$ and $(x+2)^2 = 10$. The sum of the roots each equation is -4, so the answer is $\boxed{8}$

Proposed by Satvik Lolla

Problem 5

The polynomial $x^5 - ax^3 - bx^2 + cx + d$ has five distinct roots. Four of these roots are $-7, -4, -3$, and 9. Find the value of d.

Solution

We can use Vieta's. Note that the coefficient of x^4 is 0, so the last root, r, must satisfy the equation $9 - 7 - 4 - 3 + r = 0 \implies r = 5$.

Since d is the negative product of the roots:

$$d = \boxed{3780}.$$

Proposed by Orion Foo

Problem 6

Neel has a third-degree polynomial, P. He then found $P(x), P(x+1), P(x+2), P(x+3), P(x+4)$ for some fixed x. However, he then realizes that he forgot $P(x+3)$. Suppose that $P(x) = 17, P(x+1) = 44, P(x+2) = 50, P(x+4) = 5$. If $P(x+3)$ can be written as $\frac{m}{n}$ for positive integers m, n, find $m+n$.

Solution

We can use finite difference. Note that the third difference is constant, so let that be d; then, we can use finite differences to compute that $P(x+3) = 50 + (-15 + d) = 5 - (-36 + 3d)$. From this we can compute $d = \frac{3}{2}$ and $P(x+3) = \frac{73}{2}$, so our final answer is $\boxed{75}$.

Proposed by Sumedh Vangara

Problem 7

Suppose that for positive reals x and y: $x + y = 10$. If the minimum value of $(1 + \frac{1}{x})(1 + \frac{1}{y})$ can be written as $\frac{m}{n}$ where $gcd(m,n) = 1$, find $m+n$.

Solution

Expanding $(1 + \frac{1}{x})(1 + \frac{1}{y})$, we get $1 + \frac{x+y}{xy} + \frac{1}{xy}$.

We may use the AM-GM inequality,

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$25 \geq xy$$

Plugging these values back into our original expression,

$$1+\frac{10}{25}+\frac{1}{25}=\frac{36}{25}$$

So our final answer is $\boxed{61}$.

Proposed by Orion Foo

Problem 8

Find the smallest positive integer n such that $n^4-8n^3+24n^2-32n+665$ is a perfect square.

Solution

Notice that the expression can be rewritten as

$$(n-2)^4+649$$

through the binomial theorem. Setting this equal to some k^2

$$(n-2)^4+649=k^2$$

$$649=k^2-[(n-2)^2]^2$$

$$649=(k+(n-2)^2)(k-(n-2)^2)$$

We know that both of the terms on the right are integers that multiply to 649 and have a difference of twice some perfect square. Considering the factors 1 and 649, their difference is 2×18^2, so n-2 is 18, or $n=\boxed{20}$.

You could also consider other factorizations of 649 and with negative terms, but they produce no new solutions.

Proposed by Milo Stammers

Problem 9

Define

$$f(a,b,c)=lcm(\gcd(a,b),lcm(a,c),bc)$$

.

Given that prime factorization $f(2022,9009,7)$ can be written as $p_1^{e_1}..p_k^{e_k}$, find $p_1e_1+...+p_ke_k$.

Solution

Given that some prime p goes into a, x times, b, y times, and c, z times. The power of p in f(a,b,c) would be

$$\max(\min(x,y),\max(x,z),y+z)$$

Notice that if x>y+z, the result is x, otherwise the answer is y+z.

We are trying to find

$$f(2*3*337,3^2*7*11*13,7)$$

Using our rules above, our answer would be 2*3^2*7^2*11*13*337. Thus the final answer is 2+6+14+11+13+337=$\boxed{383}$.

Proposed by Milo Stammers

Problem 10

How many ordered septuples (a,b,c,d,e,f,g) of nonnegative integers satisfy $a^4+b^4+c^4+d^4+e^4+f^4+7=40^g$?

Solution

If $g \geq 2$, take the equation mod 16; we have $16|40^g$ so $a^4+b^4+c^4+d^4+e^4+f^4 \equiv 9 \pmod{16}$. But note that fourth powers are either 0 or 1 mod 16, so this is impossible. Thur our only solutions are when $g=0,1$. Clearly we can't have $g=0$ since the LHS is at least 7. Thus, we must have $g=1$, in which case (a,b,c,d,e,f) must be some permutation of $(2,2,1,0,0,0)$. There are $\frac{6!}{2!3!}=\boxed{60}$ such permutations.

Proposed by Andrew Yuan