PVMT 2022: High School Division Algebra/Number Theory Round

Problem 1

How many digits are in the number $8^5 * 5^{10}$?

Solution

We rewrite 8^5 as 2^{15}. Our product now simplifies to $10^{10} * 2^5$. This number is equal to $3.2 * 10^{11}$, so it has $\boxed{12}$ digits.

Proposed by Orion Foo

Problem 2

In the land of Kemgod, Franklin can use each dollar to buy either 5 grams of chemical A with one dollar or 6 grams of chemical B. Let a number n be *purchasable* if Franklin can spend some amount of money to buy n total grams of chemicals. If Franklin has infinite money, what is the largest number that is not *purchasable*?

Solution

This follows from the postage-stamp theorem, which states that the largest number that is not purchasable is equal to $5 \times 6 - 5 - 6 = \boxed{19}$.

Proposed by Satvik Lolla

Problem 3

Consider the polynomial $f(x) = x^2 + 4x - 12$. Find the absolute value of the sum of all x such that $f(f(x)) = 0$.

Solution

Completing the square gives $f(x) = (x+2)^2 - 16$. $f(f(x))$ can then be written as $((x+2)^2 - 16+2)^2 - 16$. Setting this equal to 0, we have the following:

$$((x+2)^2 - 16 + 2)^2 - 16 = 0$$
$$(x+2)^2 - 14 = \pm 4$$

This gives two equations: $(x+2)^2 = 18$ and $(x+2)^2 = 10$. The sum of the roots each equation is -4, so the answer is $\boxed{8}$

Proposed by Satvik Lolla

Problem 4

Find the sum of all integers x such that $(x^2 - 6x + 11)^2 + 1$ is a prime.

Solution

An analysis of the parity of the expression will suffice.

First, we note that $x^2 - 7x = x(x-7)$, which is always an even number (odd times even = even). Hence, $(x^2 - 7x + 11)$ is odd and it follows that $(x^2 - 7x + 11)^2 + 1$ is even.

However, the if the expression has to evaluate to a prime number, the only possibility we have is if the prime number is 2. Plugging in,

$$(x^2 - 7x + 11)^2 = 1$$
$$x^2 - 7x + 11 = \pm 1$$

Solving both quadratics, we arrive at

$$(x-5)(x-2) = 0 \text{ and } (x-4)(x-3) = 0.$$
$$\implies \boxed{14}$$

Proposed by Orion Foo

Problem 5

Alice, Bob, and Charlie work together to do a job. If Alice and Bob work at their normal speed but Charlie works at twice his normal speed, then they can finish the job in 15 minutes. If Alice and Charlie work at their normal speed but Bob works at twice his normal speed, then they can finish the job in 12 minutes. If Bob and Charlie work at their normal speed but Alice works at twice her normal speed, then they can finish the job in 10 minutes. If Alice, Bob, and Charlie each work at their normal speed, then how many minutes will it take for them to complete the job?.

Solution

Let a, b, c be how many minutes it takes Alice, Bob, and Charlie to do the job, respectively. So the speeds at which they do the jobs are $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$. So we have $1/a + 1/b + 2/c = 4$, $1/a + 2/bc + 1/c = 5$, and $2/a + 1/b + 1/c = 6$; summing up, we get $1/a + 1/b + 1/c = 15/4$, so it takes them $60/(15/4) = \boxed{16}$ minutes to finish when they all work at their normal speed.

Proposed by Andrew Yuan

Problem 6

Define the sequence x_n as $x_0 = 3$, $x_1 = 7$, and $x_n = 2x_{n-1} + 3x_{n-2}$ for $n \geq 2$. To the nearest integer, what is the ratio $\frac{x_{2022}}{x_{2021}}$?

Solution

Dividing the equation by x_{n-1}, we get

$$\frac{x_n}{x_{n-1}} = 2 + 3\frac{x_{n-2}}{x_{n-1}}$$

For large values of n, the ratios will be roughly constant, equal to some value r, which we wish to find.

$$r = 2 + \frac{3}{r}$$

$$r^2 - 2r - 3 = 0$$

$$r = \frac{2+\sqrt{4+12}}{2} = \boxed{3}$$

Proposed by Milo Stammers

Problem 7

Define

$$f(a,b,c) = lcm(\gcd(a,b), lcm(a,c), bc)$$

.

Given that prime factorization $f(2022, 9009, 7)$ can be written as $p_1^{e_1}..p_k^{e_k}$, find $p_1e_1 + ... + p_ke_k$.

Solution

Given that some prime p goes into a, x times, b, y times, and c, z times. The power of p in f(a,b,c) would be

$$\max(\min(x,y), \max(x,z), y+z)$$

Notice that if x>y+z, the result is x, otherwise the answer is y+z.

We are trying to find

$$f(2*3*337, 3^2*7*11*13, 7)$$

Using our rules above, our answer would be 2*3^2*7^2*11*13*337. Thus the final answer is 2+6+14+11+13+337=$\boxed{383}$.

Proposed by Milo Stammers

Problem 8

Take an ordered-2023 tuple A, and define a function $f(A)$ that gives a new ordered-2023 tuple containing the roots to

$$a_{2022}x^{2022} + a_{2021}x^{2021} + ... + a_0$$

in some order that you may choose. Given that $f(f(X)) = (-1, -1, -1, ... -1)$ is possible, with 2022 -1's, the last element of X being -2020, and the first element of $f(X)$ being 2021, let K be the second to last element of X. Given that K can be expressed as $(pq)2^r$, where p, q are odd, find $p+q+r$.

Solution

For all the roots to be -1, the polynomial associate with $f(X)$ must be

$$k(x+1)^{2022}$$

So $f(X)$ is (2021$\binom{2022}{0}$,2021$\binom{2022}{1}$,...,2021$\binom{2022}{2022}$), because the first element must reduce to 2021. The second to last element of X divided by the first must be the negative of the sum of all the elements in $f(X)$ by Vieta's formulas. The sum of the above elements would become 2021*2^{2022}. Let x be the second to last element of X,

$$-\frac{x}{-2020} = 2021 * 2^{2022}$$

$$x = 2020 * 2021 * 2^{2022} = 505 * 2021 * 2^{2024}$$

Hence, p+q+r=505+2021+2024=$\boxed{4550}$

Proposed by Milo Stammers

Problem 9

How many ordered septuples (a,b,c,d,e,f,g) of nonnegative integers satisfy $a^4+b^4+c^4+d^4+e^4+f^4+7=40^g$?

Solution

If $g \geq 2$, take the equation mod 16; we have $16|40^g$ so $a^4+b^4+c^4+d^4+e^4+f^4 \equiv 9 \pmod{16}$. But note that fourth powers are either 0 or 1 mod 16, so this is impossible. Thur our only solutions are when $g=0,1$. Clearly we can't have $g=0$ since the LHS is at least 7. Thus, we must have $g=1$, in which case (a,b,c,d,e,f) must be some permutation of $(2,2,1,0,0,0)$. There are $\frac{6!}{2!3!} = \boxed{60}$ such permutations.

Proposed by Andrew Yuan

Problem 10

Let F_i the ith Fibonacci number (in particular $F_1=1, F_2=1, F_3=2$). Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{F_{2j-1}}{5^i}$$

can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers, find $p+q$. Binet's formula may come in handy:

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

Solution

Trying out for small values of i, you may notice the pattern that

$$\sum_{j=1}^{i} F_{2j-1} = F_{2i}$$

This can be proves through induction: For i=1, the sum is simply F_1=F_2.

Assuming $\sum_{j=1}^{i} F_{2j-1} = F_{2i}$ for some i, if we add $F_2 i+1$ to both sides and using the definition of the Fibonacci numbers $\sum_{j=1}^{i+1} F_{2j-1} = F_{2i+2}$. Our identity is thus proven by induction.

This simplifies the expression to

$$\sum_{i=1}^{\infty} \frac{F_{2i}}{5^i}$$

Using Binet's Formula and simplifying, we reduce it to two geometric series

$$\frac{1}{\sqrt{5}}[\sum_{i=1}^{\infty}(\frac{3+\sqrt{5}}{10})^i - \sum_{i=1}^{\infty}(\frac{3-\sqrt{5}}{10})^i]$$

$$\frac{1}{\sqrt{5}}[\frac{\frac{3+\sqrt{5}}{10}}{1-\frac{3+\sqrt{5}}{10}} - \frac{\frac{3-\sqrt{5}}{10}}{1-\frac{3-\sqrt{5}}{10}}]$$

With some algebra, we find that this evaluates to $\frac{5}{11}$. This gives the answer $\boxed{16}$.

Proposed by Sumedh Vangara, Solution by Milo Stammers and Sumedh Vangara