PVMT 2022: Team Round

Problem 1

A rectangular box has a volume of 64. Find the sum of the dimensions of the box with the minimum surface area.

Problem 2

Avyukth has trouble remembering his address, but knows that it is a 5 digit number such that:

1. The first two digits make a prime
2. The second and third digits make a perfect square
3. The third and fourth digits make a prime
4. The ratio of the third digit to the second is equal to the ratio of the two digit number formed by the fourth and fifth digits, to the third digit, which is equal to the first digit

What is his address?

Problem 3

In the game of *Fim*, two players take turns removing $1, 2$, or 3 rubber falcons from the center of a table. This continues until there are no more falcons left on the table. The player that takes the last falcon from the center of the table loses.

Find the largest initial number of falcons, m, where $m \leq 100$, such that the player who moves first is guaranteed to lose. Assume perfect play from both players.

Problem 4

What is the absolute value of the sum of the 2022nd powers of the roots of this polynomial:

$$x^{2022} + 3x^2 - 2x + 1$$

Problem 5

If n is the number of POSITIVE integer solutions to $a + b + c + d + e = 12$ and m is the number of NON-NEGATIVE integer solutions to $f + g + h = 15$, find $n + m$.

Problem 6

Given a factorization of 2022, with no limit on the number of factors, but none of which can be one, take the sum of the factors. What is the minimum possible value?

Problem 7

Jane and Joe are playing a game with an unfair coin. They repeatedly flip the coin until one of them wins. Jane wins if two consecutive flips are either both heads, or a tails followed by a heads. Joe wins if two consecutive flips are heads followed by tails.

Given that Jane has a 90% chance of winning, the maximum probability that the coin shows heads on a given flip can be expressed as $\frac{a}{b} + \sqrt{\frac{c}{d}}$ where a and b are relatively prime positive integers, and c and d are relatively prime positive integers. Find $a+b+c+d$.

Problem 8

Given quadrilateral ABCD with incenter I, $\overline{AB}$=10, and $\overline{CD}$=20. What is $\overline{BC}$+$\overline{DA}$?

Problem 9

The area of the graph enclosed by the function $|x-y|+|x+y|+x=1$ can be written as simplified fraction $\frac{p}{q}$. What is p+q?

Problem 10

How many ways are there for players to make moves to end a game of tic-tac-toe in a draw? Assume O always goes first. Rotations and reflections are distinct.

Problem 11

Take any convex equilateral hexagon(not necessarily regular) of side length 1. The angle bisector of one vertex and the bisector of the opposite vertex intersect at P. 6 altitudes are dropped from P to each side of the hexagon(or possibly their extensions). What is the maximum possible value of the sum of the lengths of these altitudes? The answer will be in the form $\sqrt{q}$, give us q.

Problem 12

Let $f(x)$ be a function, defined as such: $f(0)=1$, and $f(x)=2x$ for $x \neq 0$. Let $k = f(0) + f(1) + f(2) + f(3) .... + f(256)$. What is the sum of the prime factors of k?

Problem 13

Let ABC be a triangle with incenter I. Let the incircle be tangent to AB at M. Given $AM = 2 \cdot MB$, find $\frac{[AIB]}{[AIC]-[BIC]}$.

Note: $[ABC]$ denotes the area of triangle ABC.

Problem 14

John has an infinite deck of cards containing exactly four cards labeled each power of 7 (i.e. he has four 1's, four 7's, four 49's... etc). Let S denote the set of all distinct numbers John can obtain from adding together the labels of one or more of his cards (so for example, 7 is in S and so is $7+7+7+7=28$, but $7+7+7+7+7=35$ is not). Let N be the sum of the $5^{2001}-1$ smallest elements of S. What is the remainder when N is divided by 1000?

Problem 15

The complex roots of the polynomial

$$((z-(12+2i))^k-1)((z-(17-10i))^k-1)=0$$

are plotted onto the complex plane for all positive integers k. What is the smallest area of a rectangle which contains all the roots for every k?

Problem 16

Let ABC be a triangle inscribed in a circumcircle with radius 4 and with center O, and have incenter I. Let the tangents to the circumcircle at B and C intersect at K. If $\angle BAC = 60$ and the perimeter of ABC is 16, and the product of the side lengths of ABC is 128, find KI^2.

Problem 17

If the two smallest roots of $-385x^4+218x^3+144x^2+22x+1$ are a and b, then $|a+b|$ can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p+q$.

Problem 18

In a regular n-gon, with n ranging from 44 to 100 inclusive, each vertex is connected to its adjacent vertex and the center of the polygon. Two non-adjacent vertices are picked(can't be center), and are connected. This is done 22 times, with no vertex being picked more than once. What is the sum of all n such that it's always possible for a bug to choose a starting position, then move along the connections such that it crosses each connection exactly once?

Problem 19

Take some number n, and define its k-ness through the process:

1. Find n mod k, if this is 0, then add 1 to its k-ness
2. Subtract n mod k from n(reducing it to the next lowest multiple of k) then divide by k
3. If n is 0 stop, otherwise repeat

Let f_k(n) denote the k-ness of n. Find

$$\sum_{m=2}^{5}\sum_{p=1}^{7-m} f_{m^p}(1798)$$

Problem 20

What is the maximum value of the expression $|\frac{a-b}{5-\bar{a}b}|$, where a and b are complex numbers such that $|a|\leq 1, |b|\leq 1$. Given that the answer be written as $\frac{p}{q}$ in simplest form, find $p+q$.