PVMT 2022: High School Division Algebra/Number Theory Round

Problem 1

How many digits are in the number $8^5 * 5^{10}$?

Problem 2

In the land of Kemgod, Franklin can use each dollar to buy either 5 grams of chemical A with one dollar or 6 grams of chemical B. Let a number n be *purchasable* if Franklin can spend some amount of money to buy n total grams of chemicals. If Franklin has infinite money, what is the largest number that is not *purchasable*?

Problem 3

Consider the polynomial $f(x) = x^2 + 4x - 12$. Find the absolute value of the sum of all x such that $f(f(x)) = 0$.

Problem 4

Find the sum of all integers x such that $(x^2 - 6x + 11)^2 + 1$ is a prime.

Problem 5

Alice, Bob, and Charlie work together to do a job. If Alice and Bob work at their normal speed but Charlie works at twice his normal speed, then they can finish the job in 15 minutes. If Alice and Charlie work at their normal speed but Bob works at twice his normal speed, then they can finish the job in 12 minutes. If Bob and Charlie work at their normal speed but Alice works at twice her normal speed, then they can finish the job in 10 minutes. If Alice, Bob, and Charlie each work at their normal speed, then how many minutes will it take for them to complete the job?.

Problem 6

Define the sequence x_n as $x_0 = 3$, $x_1 = 7$, and $x_n = 2x_{n-1} + 3x_{n-2}$ for $n \geq 2$. To the nearest integer, what is the ratio $\frac{x_{2022}}{x_{2021}}$?

Problem 7

Define

$$f(a,b,c) = lcm(\gcd(a,b), lcm(a,c), bc)$$

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Given that prime factorization $f(2022, 9009, 7)$ can be written as $p_1^{e_1}..p_k^{e_k}$, find $p_1e_1 + ... + p_ke_k$.

Problem 8

Take an ordered-2023 tuple A, and define a function $f(A)$ that gives a new ordered-2023 tuple containing the roots to

$$a_{2022}x^{2022} + a_{2021}x^{2021} + ... + a_0$$

in some order that you may choose. Given that $f(f(X)) = (-1, -1, -1, ... - 1)$ is possible, with 2022 -1's, the last element of X being -2020, and the first element of $f(X)$ being 2021, let K be the second to last element of X. Given that K can be expressed as $(pq)2^r$, where p, q are odd, find $p+q+r$.

Problem 9

How many ordered septuples (a, b, c, d, e, f, g) of nonnegative integers satisfy $a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + 7 = 40^g$?

Problem 10

Let F_i the ith Fibonacci number (in particular $F_1 = 1, F_2 = 1, F_3 = 2$). Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{F_{2j-1}}{5^i}$$

can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers, find $p+q$. Binet's formula may come in handy:

$$F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$